

SPHERICAL NILPOTENT ORBITS IN POSITIVE CHARACTERISTIC

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ABSTRACT. Let G be a connected reductive linear algebraic group defined over an algebraically closed field of characteristic p . Assume that p is good for G . In this note we classify all the spherical nilpotent G -orbits in the Lie algebra of G . The classification is the same as in the characteristic zero case obtained by D.I. Panyushev in 1994, [32]: for e a nilpotent element in the Lie algebra of G , the G -orbit $G \cdot e$ is spherical if and only if the height of e is at most 3.

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1. INTRODUCTION

Let G be a connected reductive linear algebraic group defined over an algebraically closed field k of characteristic $p > 0$. With the exception of Subsection 4.5, we assume throughout that p is *good* for G (see Subsection 2.1 for a definition).

A *spherical* G -variety X is an (irreducible) algebraic G -variety on which a Borel subgroup B of G acts with a dense orbit. Homogeneous spherical G -varieties G/H , for H a closed subgroup of G , are of particular interest. They include flag varieties (when H is a parabolic subgroup of G) as well as symmetric spaces (when H is the fixed point subgroup of an involutive automorphism of G). We refer the reader to [5] and [6] for more information on spherical varieties and for their representation theoretic significance. These varieties enjoy a remarkable property: a Borel subgroup of G acts on a spherical G -variety only with a finite number of orbits. This fundamental result is due to M. Brion [4] and É. B. Vinberg [49] independently in characteristic 0, and to F. Knop [25, 2.6] in arbitrary characteristic.

Let $\mathfrak{g} = \text{Lie } G$ be the Lie algebra of G . The aim of this note is to classify the spherical nilpotent G -orbits in \mathfrak{g} . In case k is of characteristic zero, this classification was obtained by D.I. Panyushev in 1994 in [32]. The classification is the same in case the characteristic of k is good for G : for $e \in \mathfrak{g}$ nilpotent, $G \cdot e$ is spherical if and only if the height of e is at most 3 (Theorem 3.42). The height of e is the highest degree in the grading of \mathfrak{g} afforded by a cocharacter of G associated to e (Definition 2.26).

The methods employed by Panyushev in [32] do not apply in positive characteristic, e.g. parts of the argument are based on the concept of “stabilizers in general position”; it is unknown whether these exist generically in positive characteristic. Thus a different approach is needed to address the question in this case.

We briefly sketch the contents of the paper. In Section 2 we collect the preliminary results we require. In particular, we discuss the concepts of complexity and sphericity, and more specifically the question of complexity of homogeneous spaces. In Subsection 2.5 we recall the basic results of Kempf–Rousseau Theory and in Subsection 2.6 we recall the fundamental concepts of associated cocharacters for nilpotent elements from [22, §5] and [37]. There we also recall the grading of \mathfrak{g} afforded by a cocharacter associated to a given nilpotent element and define the notion of the height of a nilpotent element as the highest occurring degree of such a grading, Definition 2.26. The complexity of fibre bundles is discussed in Subsection 2.7 which is crucial for the sequel. In particular, in Theorem 2.33 we show that the complexity of a fixed nilpotent orbit $G \cdot e$ is given by the complexity of a smaller reductive group acting on a linear space. Precisely, let λ be a cocharacter of G that is associated to e . Then P_λ is the destabilizing parabolic subgroup $P(e)$ defined by e , in the sense of Geometric Invariant Theory. Moreover, $L = C_G(\lambda(k^*))$ is a Levi subgroup of $P(e)$. We show in Theorem 2.33 that the complexity of $G \cdot e$ equals the complexity of the action of L on the subalgebra $\bigoplus_{i \geq 2} \mathfrak{g}(i, \lambda)$ of \mathfrak{g} where the grading $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i, \lambda)$ is afforded by λ . In Subsection 2.8 we recall the concept of a weighted Dynkin diagram associated to a nilpotent orbit from [10, §5]. There we also present the classification of the parabolic subgroups P of a simple algebraic group G admitting a dense action of a Borel subgroup of a Levi subgroup of P on the unipotent radical of P from [7, Thm. 4.1]. Here we also remind the reader of the classification of the parabolic subgroups of G with an abelian unipotent radical.

In Section 3 we give the classification of the spherical nilpotent orbits in good characteristic: a nilpotent element e in \mathfrak{g} is spherical if and only if the height of e is at most 3 (Theorem

3.42). In Subsections 3.1 and 3.3 we show that orbits of height 2 are spherical and orbits of height at least 4 are not, respectively. The subsequent subsections deal with the cases of height 3 nilpotent classes. For classical groups these only occur for the orthogonal groups. For the exceptional groups the height 3 cases are handled in Subsection 3.7 with the aid of a computer programme of S.M. Goodwin.

In Section 4 we discuss some further results and some applications of the classification. In Subsection 4.1 we discuss the spherical nilpotent orbits that are distinguished and in Subsection 4.2 we extend a result of Panyushev in characteristic zero to good positive characteristic: a characterization of the spherical nilpotent orbits in terms of pairwise orthogonal simple roots, see Theorem 4.14.

In Subsection 4.3 we discuss generalizations of results from [34] and [35] to positive characteristic. In Theorem 4.18 we show that if \mathfrak{a} is an abelian ideal of \mathfrak{h} , then $G \cdot \mathfrak{a}$ is a spherical variety. In Subsection 4.4 we describe a geometric characterization of spherical orbits in simple algebraic groups from [8] and [9]. Finally, in Subsection 4.5 we very briefly touch on the issue of spherical nilpotent orbits in bad characteristic.

Thanks to the fact that a Springer isomorphism between the unipotent variety of G and the nilpotent variety of \mathfrak{g} affords a bijection between the unipotent G -classes in G and the nilpotent G -orbits in \mathfrak{g} (cf. [45]), there is an analogous classification of the spherical unipotent conjugacy classes in G .

For results on algebraic groups we refer the reader to Borel's book [2] and for information on nilpotent classes we cite Jantzen's monograph [22].

2. PRELIMINARIES

2.1. Notation. Let H be a linear algebraic group defined over an algebraically closed field k . We denote the Lie algebra of H by $\text{Lie } H$ or by \mathfrak{h} . We write H° for the identity component of H and $Z(H)$ for the centre of H . The derived subgroup of H is denoted by $\mathcal{D}H$ and we write $\text{rank } H$ for the dimension of a maximal torus of H . The unipotent radical of H is denoted by $R_u(H)$. We say that H is reductive provided H° is reductive. Let K be a subgroup of H . We write $C_H(K) = \{h \in H \mid h x h^{-1} = x \text{ for all } x \in K\}$ for the centralizer of K in H .

Suppose H acts morphically on an algebraic variety X . Then we say that X is an H -variety. Let $x \in X$. Then $H \cdot x$ denotes the H -orbit of x in X and $C_H(x) = \{h \in H \mid h \cdot x = x \text{ for all } h \in H\}$ is the stabilizer of x in H .

For $e \in \mathfrak{h}$ we denote the centralizers of e in H and \mathfrak{h} by $C_H(e) = \{h \in H \mid \text{Ad}(h)e = e\}$ and $\mathfrak{c}_{\mathfrak{h}}(e) = \{x \in \mathfrak{h} \mid [x, e] = 0\}$, respectively. For S a subset of H we write $\mathfrak{c}_{\mathfrak{h}}(S) = \{x \in \mathfrak{h} \mid \text{Ad}(s)x = x \text{ for all } s \in S\}$ for the centralizer of S in \mathfrak{h} .

Suppose G is a connected reductive algebraic group. By \mathcal{N} we denote the nilpotent cone of \mathfrak{g} . Let T be a maximal torus of G . Let $\Psi = \Psi(G, T)$ denote the set of roots of G with respect to T . Fix a Borel subgroup B of G containing T and let $\Pi = \Pi(G, T)$ be the set of simple roots of Ψ defined by B . Then $\Psi^+ = \Psi(B, T)$ is the set of positive roots of G with respect to B . For $I \subset \Pi$, we denote by P_I and L_I the *standard* parabolic and *standard* Levi subgroups of G defined by I , respectively; see [10, §2].

For $\beta \in \Psi^+$ write $\beta = \sum_{\alpha \in \Pi} c_{\alpha\beta} \alpha$ with $c_{\alpha\beta} \in \mathbb{N}_0$. A prime p is said to be *good* for G if it does not divide $c_{\alpha\beta}$ for any α and β , [48, Defn. 4.1]. Let $U = R_u(B)$ and set $\mathfrak{u} = \text{Lie } U$. For a T -stable Lie subalgebra \mathfrak{m} of \mathfrak{u} we write $\Psi(\mathfrak{m}) = \{\beta \in \Psi^+ \mid \mathfrak{g}_{\beta} \subseteq \mathfrak{m}\}$ for the set of roots of \mathfrak{m} (with respect to T).

For every root $\beta \in \Psi$ we choose a generator e_β for the corresponding root space \mathfrak{g}_β of \mathfrak{g} . Any element $e \in \mathfrak{u}$ can be uniquely written as $e = \sum_{\beta \in \Psi^+} c_\beta e_\beta$, where $c_\beta \in k$. The *support* of e is defined as $\text{supp}(e) = \{\beta \in \Psi^+ \mid c_\beta \neq 0\}$.

The variety of all Borel subgroups of G is denoted by \mathcal{B} . Note that \mathcal{B} is a single conjugacy class $\mathcal{B} = \{B^g \mid g \in G\}$. Also note the isomorphism $\mathcal{B} \cong G/B$.

Let $Y(G) = \text{Hom}(k^*, G)$ denote the set of *cocharacters* (one-parameter subgroups) of G , likewise for a closed subgroup H of G , we set $Y(H) = \text{Hom}(k^*, H)$ for the set of cocharacters of H . For $\lambda \in Y(G)$ and $g \in G$ we define $g \cdot \lambda \in Y(G)$ by $(g \cdot \lambda)(t) = g\lambda(t)g^{-1}$ for $t \in k^*$; this gives a left action of G on $Y(G)$. For $\mu \in Y(G)$ we write $C_G(\mu)$ for the centralizer of μ under this action of G which coincides with $C_G(\mu(k^*))$.

By a Levi subgroup of G we mean a Levi subgroup of a parabolic subgroup of G . The Levi subgroups of G are precisely the subgroups of G which are of the form $C_G(S)$ where S is a torus of G , [2, Thm. 20.4]. Note that for S a torus of G the group $C_G(S)$ is connected, [2, Cor. 11.12].

2.2. Complexity. Suppose the linear algebraic group H acts morphically on the (irreducible) algebraic variety X . Let B be a Borel subgroup of H . Recall that the *complexity* of X (with respect to the H -action on X) is defined as

$$\kappa_H(X) := \min_{x \in X} \text{codim}_X B \cdot x,$$

see also [6], [25], [28], [32], and [49].

Since the Borel subgroups of H are conjugate in H ([20, Thm. 21.3]), the complexity of the variety X is well-defined.

Since a Borel subgroup of H is connected, we have $\kappa_H(X) = \kappa_{H^\circ}(X)$. Thus for considering the complexity of an H -action, we may assume that H is connected.

Concerning basic properties of complexity, we refer the reader to [49, §9].

We return to the general situation of a linear algebraic group H acting on an algebraic variety X . For a Borel subgroup B of H , we define

$$\Gamma_X(B) := \{x \in X \mid \text{codim}_X B \cdot x = \kappa_H(X)\} \subseteq X.$$

Then we set

$$\Gamma_X := \bigcup_{B \in \mathcal{B}} \Gamma_X(B) \subseteq X.$$

For $x \in X$, we define

$$\Lambda_H(x) := \{B \in \mathcal{B} \mid \text{codim}_X B \cdot x = \kappa_H(X)\} \subseteq \mathcal{B}.$$

Remark 2.1. The following statements are immediate from the definitions.

- (i) If H acts transitively on X , then $\Gamma_X = X$.
- (ii) $B \in \Lambda_H(x)$ if and only if $x \in \Gamma_X(B)$.
- (iii) $\Lambda_H(x) = \emptyset$ if and only if $x \notin \Gamma_X$.

The complexity of a reducible variety can easily be determined from the complexities of its irreducible components: Since a Borel subgroup B of G is connected, it stabilizes each irreducible component of X , cf. [20, Prop. 8.2(d)]. Let $x \in \Gamma_X(B)$ and choose an irreducible component X' of X such that $x \in X'$. Then $\kappa_G(X) = \kappa_G(X') + \text{codim}_X X'$. Therefore, from now on we may assume that X is irreducible.

Next we recall the upper semi-continuity of dimension, e.g. see [20, Prop. 4.4].

Proposition 2.2. *Let $\varphi : X \rightarrow Y$ be a dominant morphism of irreducible varieties. For $x \in X$, let $\varepsilon_\varphi(x)$ be the maximal dimension of any component of $\varphi^{-1}(\varphi(x))$ passing through x . Then $\{x \in X \mid \varepsilon_\varphi(x) \geq n\}$ is closed in X , for all $n \in \mathbb{Z}$.*

Corollary 2.3. *Let X be an H -variety. The set $\{x \in X \mid \dim H \cdot x \leq n\}$ is closed in X for all $n \in \mathbb{Z}$. In particular, the union of all H -orbits of maximal dimension in X is an open subset of X .*

Lemma 2.4. *For every $B \in \mathcal{B}$, we have $\Gamma_X(B)$ is a non-empty open subset of X .*

Proof. Note that $\Gamma_X(B)$ is the union of B -orbits of maximal dimension. Thus, by Corollary 2.3, $\Gamma_X(B)$ is open in X . \square

Corollary 2.5. Γ_X is open in X .

Next we need an easy but useful lemma; the proof is elementary.

Lemma 2.6. *Let $\varphi : X \rightarrow Y$ be an H -equivariant dominant morphism of irreducible H -varieties. For $x \in X$ set $F_{\varphi(x)} = \varphi^{-1}(\varphi(x))$. Then $F_{\varphi(x)}$ is $C_H(\varphi(x))$ -stable.*

Before we can prove the main result of this subsection we need another preliminary result, see [20, Thm. 4.3].

Theorem 2.7. *Let $\varphi : X \rightarrow Y$ be a dominant morphism of irreducible varieties. Set $r = \dim X - \dim Y$. Then there is a non-empty open subset V of Y such that $V \subseteq \varphi(X)$ and if $Y' \subseteq Y$ is closed, irreducible and meets V and Z is a component of $\varphi^{-1}(Y')$ which meets $\varphi^{-1}(V)$, then $\dim Z = \dim Y' + r$. In particular, if $v \in V$, then $\dim \varphi^{-1}(v) = r$.*

For the remainder of this section let G be connected reductive. Let $\varphi : X \rightarrow Y$ be a G -equivariant dominant morphism of irreducible G -varieties. Then $\kappa_G(Y) \leq \kappa_G(X)$, [49, §9]. In the main result of this subsection we give an interpretation for the difference $\kappa_G(X) - \kappa_G(Y)$ in terms of the complexity of a smaller subgroup acting on a fibre of φ .

Theorem 2.8. *Let $\varphi : X \rightarrow Y$ be a G -equivariant dominant morphism of irreducible G -varieties. For $x \in X$ set $F_{\varphi(x)} = \varphi^{-1}(\varphi(x))$. Then for every $B \in \mathcal{B}$ there exists $x \in \Gamma_X(B)$ such that for $H = C_B(\varphi(x))^\circ$ we have*

$$\kappa_G(X) = \kappa_G(Y) + \kappa_H(Z),$$

where Z is an irreducible component of $F_{\varphi(x)}$ passing through x .

Proof. Let $B \in \mathcal{B}$. Let V be a non-empty open subset of Y which satisfies the conditions in Theorem 2.7. Since Y is irreducible, Lemma 2.4 implies that $\Gamma_Y(B) \cap V \neq \emptyset$. For $y \in \Gamma_Y(B) \cap V$, Theorem 2.7 implies that any component of $\varphi^{-1}(y)$ has dimension $r = \dim X - \dim Y$, in particular, $\dim \varphi^{-1}(y) = r$. Since $\varphi^{-1}(\Gamma_Y(B) \cap V)$ is open in X , we have $\varphi^{-1}(\Gamma_Y(B) \cap V) \cap \Gamma_X(B) \neq \emptyset$, by Lemma 2.4. Now choose $x \in \varphi^{-1}(\Gamma_Y(B) \cap V) \cap \Gamma_X(B)$. In particular, $\dim F_{\varphi(x)} = r$. Lemma 2.6 implies that $F_{\varphi(x)}$ is $C_B(\varphi(x))$ -stable. Clearly, $C_B(x)$ is the stabilizer of x in $C_B(\varphi(x))$. Thus we obtain

$$\begin{aligned}
\text{codim}_{F_{\varphi(x)}} C_B(\varphi(x)) \cdot x &= \dim F_{\varphi(x)} - \dim C_B(\varphi(x)) \cdot x \\
&= r - \dim C_B(\varphi(x)) + \dim C_B(x) \\
&= \dim X - \dim Y - \dim C_B(\varphi(x)) + \dim C_B(x) + \dim B - \dim B \\
&= (\dim X - \dim B + \dim C_B(x)) \\
&\quad - (\dim Y - \dim B + \dim C_B(\varphi(x))) \\
&= \kappa_G(X) - \kappa_G(Y),
\end{aligned}$$

where the last equality holds because $x \in \Gamma_X(B)$ and $\varphi(x) \in \Gamma_Y(B)$.

Let Z be an irreducible component of $F_{\varphi(x)}$ which passes through x . Theorem 2.7 implies that Z has the same dimension as $F_{\varphi(x)}$. The connected group $H = C_B(\varphi(x))^\circ$ stabilizes Z . Note that for each $z \in Z$ we have $\varphi(z) = \varphi(x)$ and $C_B(z) = C_{C_B(\varphi(x))}(z)$ (observed for $z = x$ above). Since $x \in \Gamma_X(B)$, $\dim C_B(x)$ is minimal among groups of the form $C_B(z)$ for $z \in Z$. Therefore, because $C_B(z) = C_{C_B(\varphi(x))}(z)$, we see that $\dim C_{C_B(\varphi(x))}(x)$ is minimal among groups of the form $C_{C_B(\varphi(z))}(z)$ for $z \in Z$. We deduce that $x \in \Gamma_Z(H)$. Consequently,

$$\kappa_H(Z) = \dim Z - \dim C_B(\varphi(x))^\circ + \dim C_{C_B(\varphi(x))^\circ}(x) = \text{codim}_{F_{\varphi(x)}} C_B(\varphi(x)) \cdot x.$$

The result follows. \square

2.3. Spherical Varieties. A G -variety X is called *spherical* if a Borel subgroup of G acts on X with a dense orbit, that is $\kappa_G(X) = 0$. We recall some standard facts concerning spherical varieties, see [6], [25] and [32].

First we recall an important result due to É.B. Vinberg [49] and M. Brion [4] independently in characteristic zero and F. Knop [25, Cor. 2.6] in arbitrary characteristic. Let B be a Borel subgroup of G .

Theorem 2.9. *A spherical G -variety consists only of a finite number of B -orbits.*

We have an immediate corollary.

Corollary 2.10. *The following are equivalent.*

- (i) *The G -variety X is spherical.*
- (ii) *There is an open B -orbit in X .*
- (iii) *The number of B -orbits in X is finite.*

2.4. Homogeneous Spaces. Let H be a closed subgroup of G . Since G/H is a G -variety, we may consider the complexity $\kappa_G(G/H)$. Let B be a Borel subgroup of G . The orbits of B on G/H are in bijection with the (B, H) -double cosets of G . We have that $\kappa_G(G/H) = \text{codim}_{G/H} BgH/H$ for $gH \in \Gamma_{G/H}(B)$. Clearly, G acts transitively on G/H , so Remark 2.1(i) implies that we can choose a Borel subgroup B such that $B \in \Lambda_G(1H)$. Thus, for this choice of B , we have

$$\begin{aligned}
\kappa_G(G/H) &= \text{codim}_{G/H} BH/H = \dim G/H - \dim BH/H \\
(2.11) \quad &= \dim G/H - \dim B/B \cap H \\
&= \dim G - \dim H - \dim B + \dim B \cap H.
\end{aligned}$$

Following M. Krämer [26], a subgroup H of G is called *spherical* if $\kappa_G(G/H) = 0$.

Since $\kappa_G(G/H) = \kappa_G(G/H^\circ)$, by (2.11), in considering the complexity of homogeneous spaces G/H we may assume that the subgroup H is connected.

We have an easy lemma.

Lemma 2.12. *Let G be connected reductive and let H be a subgroup of G which contains the unipotent radical of a Borel subgroup of G . Then H is spherical. In particular, a parabolic subgroup of G is spherical.*

Proof. Let B be a Borel subgroup of G such that $U = R_u(B) \leq H$. Denote by B^- the opposite Borel subgroup to B , relative to some maximal torus of B , see [20, §26.2 Cor. C]. The *big cell* B^-U is an open subset of G , [20, Prop. 28.5]. We have $B^-U \subseteq B^-H$, so B^-H is a dense subset of G . Thus, G/H is spherical. \square

Remark 2.13. If both G and H are reductive, then G/H is an affine variety, see [39, Thm. A]. This case has been studied greatly. The classification of spherical reductive subgroups of the simple algebraic groups in characteristic zero was obtained by M. Krämer [26] and was shown to be the same in positive characteristic by J. Brundan [7]. M. Brion [5] classifies all the spherical reductive subgroups of an arbitrary reductive group in characteristic zero. In positive characteristic no such classification is known. However, the classification of the reductive spherical subgroups in simple algebraic groups in positive characteristic follows from work of T.A. Springer [46] (see also G. Seitz [44]), J. Brundan [7] and R. Lawther [27].

Important examples of reductive spherical subgroups are centralizers of involutive automorphisms of G : Suppose that $\text{char } k \neq 2$ and let θ be an involutive automorphism of G . Then the fixed point subgroup $C_G(\theta) = \{g \in G \mid \theta(g) = g\}$ of G is spherical, see [46, Cor. 4.3.1].

For more on the complexity and sphericity of homogeneous spaces see [4], [28] and [31].

Remark 2.14. In order to compute the complexity of an orbit variety, it suffices to determine the complexity of a homogeneous space. For, suppose that G acts on an algebraic variety X . Let $x \in X$. Since G is connected, the orbit $G \cdot x$ is irreducible. The map $\pi_x : G/C_G(x) \rightarrow G \cdot x$, by $\pi_x(gC_G(x)) = g \cdot x$ is a bijective G -equivariant morphism, [22, §2.1]. Thus, by applying Theorem 2.8 to π_x , we have

$$(2.15) \quad \kappa_G(G/C_G(x)) = \kappa_G(G \cdot x).$$

The relevance of (2.15) is that the left hand side is easier to compute, since calculating $\kappa_G(G/C_G(x))$ only requires the study of groups of the form $C_B(x)$, cf. (2.11), where B is a Borel subgroup of G .

2.5. Kempf–Rousseau Theory. Next we require some standard facts from Geometric Invariant Theory, see [24], also see [37, §2], [40, §7]. Let X be an affine variety and $\phi : k^* \rightarrow X$ be a morphism of algebraic varieties. We say that $\lim_{t \rightarrow 0} \phi(t)$ exists if there exists a morphism $\widehat{\phi} : k \rightarrow X$ such that $\widehat{\phi}|_{k^*} = \phi$. If such a limit exists, we set $\lim_{t \rightarrow 0} \phi(t) = \widehat{\phi}(0)$. Note, that if such a morphism $\widehat{\phi}$ exists, it is necessarily unique.

Let λ be a cocharacter of G . Define $P_\lambda = \{x \in G \mid \lim_{t \rightarrow 0} \lambda(t)x\lambda(t)^{-1} \text{ exists}\}$. Then P_λ is a parabolic subgroup of G , the unipotent radical of P_λ is given by $R_u(P_\lambda) = \{x \in G \mid \lim_{t \rightarrow 0} \lambda(t)x\lambda(t)^{-1} = 1\}$, and a Levi subgroup of P_λ is the centralizer $G_G(\lambda) = C_G(\lambda(k^*))$ of the image of λ in G , [47, §8.4].

Let the connected reductive group G act on the affine variety X and suppose $x \in X$ is a point such that $G \cdot x$ is not closed in X . Let C denote the unique closed G -orbit in the closure of $G \cdot x$, cf. [39, Lem. 1.4]. Set $\Lambda(x) := \{\lambda \in Y(G) \mid \lim_{t \rightarrow 0} \lambda(t) \cdot x \text{ exists and lies in } C\}$. Then there is a so-called *optimal class* $\Omega(x) \subseteq \Lambda(x)$ of cocharacters associated to x . The following theorem is due to G.R. Kempf, [24, Thm. 3.4] (see also [43]).

Theorem 2.16. *Assume as above. Then we have the following:*

- (i) $\Omega(x) \neq \emptyset$.
- (ii) *There exists a parabolic subgroup $P(x)$ of G such that $P(x) = P_\lambda$ for every $\lambda \in \Omega(x)$.*
- (iii) $\Omega(x)$ *is a single $P(x)$ -orbit.*
- (iv) *For $g \in G$, we have $\Omega(g \cdot x) = g \cdot \Omega(x)$ and $P(g \cdot x) = gP(x)g^{-1}$. In particular, $C_G(x) \leq N_G(P(x)) = P(x)$.*

Frequently, $P(x)$ in Theorem 2.16 is called the *destabilizing* parabolic subgroup of G defined by $x \in X$.

2.6. Associated Cocharacters. In this subsection we closely follow A. Premet [37]; also see [22, §5]. We recall that p is a good prime for G throughout this section.

Every cocharacter $\lambda \in Y(G)$ induces a grading of \mathfrak{g} :

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i, \lambda),$$

where

$$\mathfrak{g}(i, \lambda) = \{x \in \mathfrak{g} \mid \text{Ad}(\lambda(t))(x) = t^i x \text{ for all } t \in k^*\},$$

see [22, §5.1]. For P_λ as in the previous subsection, we have the following equalities: $\text{Lie } P_\lambda = \bigoplus_{i \geq 0} \mathfrak{g}(i, \lambda)$; $\text{Lie } R_u(P_\lambda) = \bigoplus_{i > 0} \mathfrak{g}(i, \lambda)$; and $\text{Lie } C_G(\lambda) = \mathfrak{g}(0, \lambda)$. Frequently, we write $\mathfrak{g}(i)$ for $\mathfrak{g}(i, \lambda)$ once we have fixed a cocharacter $\lambda \in Y(G)$.

Let H be a connected reductive subgroup of G . A nilpotent element $e \in \mathfrak{h}$ is called *distinguished in \mathfrak{h}* provided each torus in $C_H(e)$ is contained in the centre of H , [22, §4.1].

The following characterization of distinguished nilpotent elements in the Lie algebra of a Levi subgroup of G can be found in [22, §4.6, §4.7].

Proposition 2.17. *Let $e \in \mathfrak{g}$ be nilpotent and let L be a Levi subgroup of G . Then e is distinguished in $\text{Lie } L$ if and only if $L = C_G(S)$, where S is a maximal torus of $C_G(e)$.*

Next we recall the definition of an associated cocharacter, see [22, §5.3].

Definition 2.18. A cocharacter $\lambda : k^* \rightarrow G$ is *associated* to $e \in \mathcal{N}$ if $e \in \mathfrak{g}(2, \lambda)$ and there exists a Levi subgroup L of G such that e is distinguished in $\text{Lie } L$, and $\lambda(k^*) \leq \mathcal{D}L$.

Remark 2.19. In view of Proposition 2.17, the last two conditions in Definition 2.18 are equivalent to the existence of a maximal torus S of $C_G(e)$ such that $\lambda(k^*) \leq \mathcal{D}C_G(S)$. We will use this fact frequently in the sequel.

Let $e \in \mathcal{N}$. In [37, §2.4, Prop. 2.5], A. Premet explicitly defines a cocharacter of G which is associated to e . Moreover, in [37, Thm. 2.3], Premet shows that each of these associated cocharacters belongs to the optimal class $\Omega(e)$ determined by e . Premet shows this under the so called *standard hypotheses* on G , see [22, §2.9]. These restrictions were subsequently removed by G. McNinch in [29, Prop. 16] so that this fact holds for any connected reductive

group G in good characteristic. It thus follows from [29, Prop. 16], Theorem 2.16(iv), and the fact that any two associated cocharacters are conjugate under $C_G(e)$, [22, Lem. 5.3], that all the cocharacters of G associated to $e \in \mathcal{N}$ belong to the optimal class $\Omega(e)$ defined by e ; see also [29, Prop. 18, Thm. 21]. This motivates and justifies the following notation which we use in the sequel.

Definition 2.20. Let $e \in \mathfrak{g}$ be nilpotent. Then we denote the set of cocharacters of G associated to e by

$$\Omega_G^a(e) := \{\lambda \in Y(G) \mid \lambda \text{ is associated to } e\} \subseteq \Omega(e).$$

Further, if H is a (connected) reductive subgroup of G with $e \in \mathfrak{h}$ nilpotent we also write $\Omega_H^a(e)$ to denote the cocharacters of H that are associated to e .

As indicated above, in good characteristic, associated cocharacters are known to exist for any nilpotent element $e \in \mathfrak{g}$; more precisely, we have the following, [22, §5.3]:

Proposition 2.21. *Suppose that p is good for G . Let $e \in \mathfrak{g}$ be nilpotent. Then $\Omega_G^a(e) \neq \emptyset$. Moreover, if $\lambda \in \Omega_G^a(e)$ and $\mu \in Y(G)$, then $\mu \in \Omega_G^a(e)$ if and only if μ and λ are conjugate by an element of $C_G(e)$.*

Fix a nilpotent element $e \in \mathfrak{g}$ and an associated cocharacter $\lambda \in \Omega_G^a(e)$ of G . Set $P = P_\lambda$. By Theorem 2.16(ii), P only depends on e and not on the choice of the associated cocharacter λ . Note that $C_G(\lambda)$ stabilizes $\mathfrak{g}(i)$ for every $i \in \mathbb{Z}$. For $n \in \mathbb{Z}_{\geq 0}$ we set

$$\mathfrak{g}_{\geq n} = \bigoplus_{i \geq n} \mathfrak{g}(i) \quad \text{and} \quad \mathfrak{g}_{> n} = \bigoplus_{i > n} \mathfrak{g}(i).$$

Then we have

$$\mathfrak{g}_{\geq 0} = \text{Lie } P \quad \text{and} \quad \mathfrak{g}_{> 0} = \text{Lie } R_u(P).$$

Also, $C_G(e) = C_P(e)$, by Theorem 2.16(iv).

The next result is [22, Prop. 5.9(c)].

Proposition 2.22. *The P -orbit of e in $\mathfrak{g}_{\geq 2}$ is dense in $\mathfrak{g}_{\geq 2}$.*

Corollary 2.23. *The $C_G(\lambda)$ -orbit of e in $\mathfrak{g}(2)$ is dense in $\mathfrak{g}(2)$.*

Define

$$C_G(e, \lambda) := C_G(e) \cap C_G(\lambda).$$

Corollary 2.24. *Let $e \in \mathcal{N}$. Then*

- (i) $\dim C_G(e) = \dim \mathfrak{g}(0) + \dim \mathfrak{g}(1)$;
- (ii) $\dim R_u(C_G(e)) = \dim \mathfrak{g}(1) + \dim \mathfrak{g}(2)$;
- (iii) $\dim C_G(e, \lambda) = \dim \mathfrak{g}(0) - \dim \mathfrak{g}(2)$.

Proof. As $C_G(e) = C_P(e)$, part (i) is immediate from Proposition 2.22. Using the fact that $(\text{Ad}(R_u(P)) - 1)(e) \subseteq \mathfrak{g}_{\geq 3}$ (e.g. see [22, §5.10]) and Proposition 2.22, we see that $\dim \text{Ad}(R_u(P))(e) = \dim \mathfrak{g}_{\geq 3}$ and so $\dim C_{R_u(P)}(e) = \dim \mathfrak{g}(1) + \dim \mathfrak{g}(2)$. Finally, part (iii) follows from the first two. \square

The following basic result regarding the structure of $C_G(e)$ can be found in [37, Thm. A].

Proposition 2.25. *If $\text{char } k$ is good for G , then $C_G(e)$ is the semi-direct product of $C_G(e, \lambda)$ and $C_G(e) \cap R_u(P)$. Moreover, $C_G(e, \lambda)^\circ$ is reductive and $C_G(e) \cap R_u(P)$ is the unipotent radical of $C_G(e)$.*

Definition 2.26. Let $e \in \mathfrak{g}$ be nilpotent. The *height* of e with respect to an associated cocharacter $\lambda \in \Omega_G^a(e)$ is defined to be

$$\text{ht}(e) := \max_{i \in \mathbb{N}} \{i \mid \mathfrak{g}(i, \lambda) \neq 0\}.$$

Thanks to Proposition 2.21, the height of e does not depend on the choice of $\lambda \in \Omega_G^a(e)$. Since conjugate nilpotent elements have the same height, we may speak of the height of a given nilpotent orbit. Since $\lambda \in \Omega_G^a(e)$, we have $\text{ht}(e) \geq 2$ for any nilpotent element $e \in \mathfrak{g}$, cf. Definition 2.18.

Let \mathfrak{g} be classical with natural module V . Set $n = \dim V$. We write a partition π of n in one of the following two ways, either $\pi = (d_1, d_2, \dots, d_r)$ with $d_1 \geq d_2 \geq \dots \geq d_r \geq 0$ and $\sum_{i=1}^r d_i = n$; or $\pi = [1^{r_1}, 2^{r_2}, \dots]$ with $\sum_i i r_i = n$. These two notations are related by $r_i = |\{j \mid d_j = i\}|$ for $i \geq 1$.

For \mathfrak{g} classical with natural module V it is straightforward to determine the height of a nilpotent orbit from the corresponding partition of $\dim V$. We leave the proof of the next proposition to the reader.

Proposition 2.27. *Let $e \in \mathfrak{g}$ be nilpotent with partition $\pi_e = (d_1, d_2, \dots, d_r)$.*

- (i) *If $\mathfrak{g} = \mathfrak{gl}(V)$, $\mathfrak{sl}(V)$ or $\mathfrak{sp}(V)$, then $\text{ht}(e) = 2(d_1 - 1)$.*
- (ii) *If $\mathfrak{g} = \mathfrak{so}(V)$, then $\text{ht}(e) = \begin{cases} 2(d_1 - 1) & \text{if } d_1 = d_2, \\ 2d_1 - 3 & \text{if } d_1 = d_2 + 1, \\ 2(d_1 - 2) & \text{if } d_1 > d_2 + 1. \end{cases}$*

Remarks 2.28. (i). For $\text{char } k = 0$, Proposition 2.27 was proved in [33, Thm. 2.3].

(ii). If e is a nilpotent element in $\mathfrak{gl}(V)$, $\mathfrak{sl}(V)$ or $\mathfrak{sp}(V)$, then $\text{ht}(e)$ is even. If e is a nilpotent element in $\mathfrak{so}(V)$, then $\text{ht}(e)$ is odd if and only if $d_2 = d_1 - 1$.

2.7. Fibre Bundles. Let H be a closed subgroup of G . Suppose that H acts on an affine variety Y . Define a morphic action of H on the affine variety $G \times Y$ by $h \cdot (g, y) = (gh, h^{-1} \cdot y)$ for $h \in H, g \in G$ and $y \in Y$. Since H acts fixed point freely on $G \times Y$, every H -orbit in $G \times Y$ has dimension $\dim H$. There exists a surjective quotient morphism $\rho : G \times Y \rightarrow (G \times Y)/H$, [30, §1.2], [36, §4.8]. We denote the quotient $(G \times Y)/H$ by $G *_H Y$, the *fibre bundle* associated to the *principal bundle* $\pi : G \rightarrow G/H$ defined by $\pi(g) = gH$ and *fibre* Y . We denote the element $(g, y)H$ of $G *_H Y$ simply by $g * y$, see [38, §2]. Let X be a G -variety and $Y \subseteq X$ be an H -subvariety. The *collapsing* of the fibre bundle $G *_H Y$ is the morphism $G *_H Y \rightarrow G \cdot Y \subseteq X$ defined by $g * y \rightarrow g \cdot y$.

Define an action of G on $G *_H Y$ by $g \cdot (g' * y) = (gg') * y$ for $g, g' \in G$ and $y \in Y$. We then have a G -equivariant surjective morphism $\varphi : G *_H Y \rightarrow G/H$ by $\varphi(g * y) = gH$. Note that $\varphi^{-1}(gH) \cong Y$ for all $gH \in G/H$.

Proposition 2.29. *Let H be a closed subgroup of G and let Y be an H -variety. Suppose that B is a Borel subgroup of G such that $\dim B \cap H$ is minimal (among all subgroups of the form $B' \cap H$ for B' ranging over \mathcal{B}). Then we have*

$$\kappa_G(G *_H Y) = \kappa_G(G/H) + \kappa_{B \cap H}(Y).$$

Proof. We apply Theorem 2.8 to the morphism $\varphi : G *_H Y \rightarrow G/H$. Thus, for a Borel subgroup B of G and $g*y \in \Gamma_{G *_H Y}(B)$, we have that $\kappa_G(G *_H Y) = \kappa_G(G/H) + \kappa_K(Z)$, where Z is an irreducible component of $\varphi^{-1}(\varphi(g*y))$ passing through $g*y$, $K = C_B(gH)^\circ$. Note that $C_B(gH) = B \cap gHg^{-1}$. So, since $g*y \in \Gamma_{G *_H Y}(B)$, the dimension of $g^{-1}C_B(gH)g = g^{-1}Bg \cap H$ is minimal. Now, as $G *_H Y$ is a fibre bundle, for $x \in G$ we have $Y_x := \varphi^{-1}(\varphi(x*y)) \cong Y$. Define a morphism $\phi : Y_x \rightarrow Y$ by $\phi(g*y) = x^{-1}g*y$. Clearly, $xhx^{-1} \in B \cap xHx^{-1}$ acts on $g*y \in Y_x$, as $xhx^{-1} \cdot (g*y) = xhx^{-1}g*y$. Since $g = xh'$ for some $h' \in H$, we have $xhx^{-1} \cdot (g*y) = xhh'*y$. So $\phi(xhh'*y) = hh'*y$. Thus, if we define an action of $B \cap xHx^{-1}$ on Y by $xhx^{-1} \cdot y = h \cdot y$, the morphism $\phi : Y_x \rightarrow Y$ becomes a $(B \cap xHx^{-1})$ -equivariant isomorphism. It follows that $\kappa_{B \cap xHx^{-1}}(Y_x) = \kappa_{B \cap xHx^{-1}}(Y)$. Since $x^{-1}(B \cap xHx^{-1})x = x^{-1}Bx \cap H$, we finally get $\kappa_{B \cap xHx^{-1}}(Y) = \kappa_{x^{-1}Bx \cap H}(Y)$. The result follows. \square

Next we need a technical lemma.

Lemma 2.30. *Let P be a parabolic subgroup of G . Then for B ranging over \mathcal{B} , the intersection $B \cap P$ is minimal if and only if $B \cap P$ is a Borel subgroup of a Levi subgroup of P .*

Proof. We may choose a Borel subgroup B of G so that BP is open dense in G , cf. the proof of Lemma 2.12. Then the P -orbit of the base point in $G/B \cong \mathcal{B}$ is open dense in \mathcal{B} . Consequently, the stabilizer of this base point in P , that is $P \cap B$ is minimal among all the isotropy subgroups $P \cap B'$ for B' in \mathcal{B} . Clearly, B is opposite to a Borel subgroup of G contained in P . Thanks to [2, Cor. 14.13], $P \cap B$ contains a maximal torus T of G . Let L be the unique Levi subgroup of P containing T . Then [10, Thm. 2.8.7] implies that $P \cap B = T(R_u(B) \cap L)$. Clearly, $T(R_u(B) \cap L)$ is solvable and thus lies in a Borel subgroup of L . A simple dimension counting argument, using Theorem 2.7 applied to the multiplication map $B \times P \rightarrow BP$ and the fact that $\dim BP = \dim G$, shows that $P \cap B$ is a Borel subgroup of L .

Reversing the argument in the previous paragraph shows that if $P \cap B$ is a Borel subgroup of L , then BP is dense in G and thus $P \cap B$ is minimal again in the sense of the statement. \square

Next we consider a special case of Proposition 2.29.

Lemma 2.31. *Let P be a parabolic subgroup of G and let Y be a P -variety. Then*

$$\kappa_G(G *_P Y) = \kappa_L(Y),$$

where L is a Levi subgroup of P .

Proof. Proposition 2.29 implies that $\kappa_G(G *_P Y) = \kappa_G(G/P) + \kappa_{B \cap P}(Y)$, where $\dim B \cap P$ is minimal. Lemmas 2.12 and 2.30 imply that $\kappa_G(G/P) = 0$ and $B \cap P$ is a Borel subgroup of a Levi subgroup of P . The result follows. \square

Let $e \in \mathcal{N}$ be a non-zero nilpotent element, $\lambda \in \Omega_G^a(e)$ be an associated cocharacter of e and $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$ be the grading of \mathfrak{g} induced by λ . Also let P be the destabilizing parabolic subgroup of G defined by e , cf. Subsection 2.5. In particular, we have $\text{Lie } P = \mathfrak{g}_{\geq 0}$, see Subsection 2.6.

Lemma 2.32. *Let $e \in \mathcal{N}$. Then $G \cdot \mathfrak{g}_{\geq 2} = \overline{G \cdot e}$. In particular, $\dim G \cdot \mathfrak{g}_{\geq 2} = \dim G \cdot e$.*

Proof. Since $\mathfrak{g}_{\geq 2}$ is P -stable, $G \cdot \mathfrak{g}_{\geq 2}$ is closed, [21, Prop. 0.15]. Thus, since $e \in \mathfrak{g}(2) \subseteq \mathfrak{g}_{\geq 2}$, we have $\overline{G \cdot e} \subseteq G \cdot \mathfrak{g}_{\geq 2}$. By Proposition 2.22, $\overline{P \cdot e} = \mathfrak{g}_{\geq 2}$. Since $\overline{P \cdot e} \subseteq \overline{G \cdot e}$, we thus have $\mathfrak{g}_{\geq 2} \subseteq \overline{G \cdot e}$. Finally, as $\overline{G \cdot e}$ is G -stable, $G \cdot \mathfrak{g}_{\geq 2} \subseteq \overline{G \cdot e}$. The result follows. \square

Theorem 2.33. *Let $e \in \mathcal{N}$. Then*

$$\kappa_G(G \cdot e) = \kappa_L(\mathfrak{g}_{\geq 2}),$$

where L is a Levi subgroup of P .

Proof. We have $\kappa_G(G \cdot e) = \kappa_G(G/C_G(e)) = \kappa_G(G/C_P(e))$, thanks to (2.15) and the fact that $G_G(e) = C_P(e)$. Moreover, since $G *_P P/C_P(e) \cong G/C_P(e)$, it follows from Lemma 2.31 that $\kappa_G(G/C_P(e)) = \kappa_L(P/C_P(e))$. Finally, thanks to Proposition 2.22 and (2.15), we obtain $\kappa_L(P/C_P(e)) = \kappa_L(\mathfrak{g}_{\geq 2})$. The result follows. \square

Remark 2.34. For $\text{char } k = 0$, Theorem 2.33 was proved by Panyushev in [33, Thm. 4.2.2].

Remark 2.35. Thanks to Theorem 2.33, in order to determine whether a nilpotent orbit is spherical, it suffices to show that a Borel subgroup of a Levi subgroup of P acts on $\mathfrak{g}_{\geq 2}$ with a dense orbit. In our classification we pursue this approach.

2.8. Borel Subgroups of Levi Subgroups Acting on Unipotent Radicals. Let $e \in \mathfrak{g}$ be a non-zero nilpotent element and let $\lambda \in \Omega_G^a(e)$ be an associated cocharacter for e . Let $P = P_\lambda$ be the destabilizing parabolic subgroup defined by e . We denote the Levi subgroup $C_G(\lambda)$ of P by L . Our next result is taken from [22, §3]. We only consider the case when G is simple, the extension to the case when G is reductive is straightforward.

Proposition 2.36. *Let G be a simple classical algebraic group and $0 \neq e \in \mathfrak{g}$ be nilpotent with corresponding partition $\pi_e = [1^{r_1}, 2^{r_2}, 3^{r_3}, \dots]$. Let $a_i, b_i, s, t \in \mathbb{Z}_{\geq 0}$ such that $a_i + 1 = \sum_{j \geq i} r_{2j+1}$, $b_i + 1 = \sum_{j \geq i} r_{2j}$, $2s = \sum_{j \geq 0} r_{2j+1}$, and $2t + 1 = \sum_{j \geq 0} r_{2j+1}$. Then the structure of \mathcal{DL} is as follows.*

- (i) *If G is of type A_n , then \mathcal{DL} is of type $\prod_{i \geq 0} A_{a_i} \times \prod_{i \geq 1} A_{b_i}$.*
- (ii) *If G is of type B_n , then \mathcal{DL} is of type $\prod_{i \geq 1} A_{a_i} \times \prod_{i \geq 1} A_{b_i} \times B_t$.*
- (iii) *If G is of type C_n , then \mathcal{DL} is of type $\prod_{i \geq 1} A_{a_i} \times \prod_{i \geq 1} A_{b_i} \times C_s$.*
- (iv) *If G is of type D_n , then \mathcal{DL} is of type $\prod_{i \geq 1} A_{a_i} \times \prod_{i \geq 1} A_{b_i} \times D_s$.*

We use the conventions that $A_0 = B_0 = C_0 = D_0 = \{1\}$, $D_1 \cong k^*$ and $D_2 = A_1 \times A_1$.

In order to describe the Levi subgroups $C_G(\lambda)$ for the exceptional groups we need to know more about associated cocharacters. Let T be a maximal torus of G such that $\lambda(k^*) \leq T$. Now let $G_{\mathbb{C}}$ be the simple, simply connected group over \mathbb{C} with the same root system as G . Let $\mathfrak{g}_{\mathbb{C}}$ be the Lie algebra of $G_{\mathbb{C}}$. For a nilpotent element $e \in \mathfrak{g}_{\mathbb{C}}$ we can find an \mathfrak{sl}_2 -triple containing e . Let $h \in \mathfrak{g}_{\mathbb{C}}$ be the semisimple element of this \mathfrak{sl}_2 -triple. Note that h is the image of 1 under the differential of $\lambda_{\mathbb{C}} \in G_{\mathbb{C}}$ (corresponding to λ) at 1. Then there exists a set of simple roots Π of Ψ such that $\alpha(h) \geq 0$ for all $\alpha \in \Psi^+$ and $\alpha(h) = m_{\alpha} \in \{0, 1, 2\}$ for all $\alpha \in \Pi$, see [10, §5.6]. For each simple root $\alpha \in \Pi$ we attach the numerical label m_{α} to the corresponding node of the Dynkin diagram. The resulting labels form the *weighted Dynkin diagram* $\Delta(e)$ of e . We denote the set of weighted Dynkin diagrams of G by $\mathcal{D}(\Pi)$. For $e, e' \in \mathfrak{g}_{\mathbb{C}}$ nilpotent, we have that $\Delta(e) = \Delta(e')$ if and only if e and e' are in the same $G_{\mathbb{C}}$ -orbit.

In order to determine the weighted Dynkin diagram of a given nilpotent orbit we refer to the method outlined in [10, §13] for the classical groups, and to the tables in *loc. cit.* for the exceptional groups.

We return to the case when the characteristic of k is good for G . In this case the classification of the nilpotent orbits does not depend on the field k . [10, §5.11]. Recently, in [37] Premet gave a proof of this fact for the unipotent classes of G which is free from case by case considerations. This applies in our case, since the classification of the unipotent conjugacy classes in G and of the nilpotent orbits in \mathcal{N} is the same in good characteristic, [10, §9 and §11]. First assume that G is simply connected and that G admits a finite-dimensional rational representation such that the trace form on \mathfrak{g} is non-degenerate; see [37, §2.3] for the motivation of these assumptions. Under these assumptions, given $\Delta \in \mathcal{D}(\Pi)$, there exists a cocharacter $\lambda = \lambda_\Delta$ of G which is associated to e , where e lies in the dense L -orbit in $\mathfrak{g}(2, \lambda)$, for $L = C_G(\lambda)$, such that

$$(2.37) \quad \text{Ad}(\lambda(t))(e_{\pm\alpha}) = t^{\pm m_\alpha} e_{\pm\alpha} \text{ and } \text{Ad}(\lambda(t))(x) = x$$

for all $\alpha \in \Pi$, $e_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$, $x \in \mathfrak{t}$ and $t \in k^*$, [37, §2.4]. We extend this action linearly to all of \mathfrak{g} . Now return to the general simple case. Let \widehat{G} be the simple, simply connected group with the same root datum as G . Then there exists a surjective central isogeny $\pi : \widehat{G} \rightarrow G$, [10, §1.11]. Also, an associated cocharacter for $e = d\pi(\widehat{e})$ in \mathfrak{g} is of the form $\pi \circ \widehat{\lambda}$, where $\widehat{\lambda}$ is a cocharacter of \widehat{G} that is associated to \widehat{e} in $\widehat{\mathfrak{g}}$. This implies that (2.37) holds for an arbitrary simple algebraic group, when the characteristic of k is good for G .

After these deliberations we can use the tables in [10, §13] to determine the structure of the Levi subgroup $C_G(\lambda)$ for the exceptional groups. Recall that $\text{Lie } C_G(\lambda) = \mathfrak{g}(0)$ and $\mathfrak{g}(0)$ is the sum of the root spaces \mathfrak{g}_α , where $\alpha \in \Psi$ with $\langle \alpha, \lambda \rangle = 0$. Let $\Pi_0 = \{\alpha \in \Pi \mid m_\alpha = 0\}$, the set of nodes α of the corresponding weighted Dynkin diagram with label $m_\alpha = 0$. Then $C_G(\lambda) = \langle T, U_{\pm\alpha} \mid \alpha \in \Pi_0 \rangle$.

It is straightforward to determine the height of a nilpotent orbit from its associated weighted Dynkin diagram. Let $\tilde{\alpha} = \sum_{\alpha \in \Pi} c_\alpha \alpha$ be the highest root of Ψ . For each simple root $\alpha \in \Pi$ we have $\mathfrak{g}_\alpha \subseteq \mathfrak{g}(m_\alpha)$ where m_α is the corresponding numerical label on the weighted Dynkin diagram, by (2.37).

Lemma 2.38. *Let $\tilde{\alpha}$ be the highest root of Ψ and set $d = \text{ht}(e)$. Then $\mathfrak{g}_{\tilde{\alpha}} \subseteq \mathfrak{g}(d)$.*

Proof. Clearly, we have $\mathfrak{g}_{\tilde{\alpha}} \subseteq \mathfrak{g}(i)$ for some $i \geq 0$. The lemma is immediate, because if $\tilde{\alpha} = \sum_{\alpha \in \Pi} c_\alpha \alpha$ and $\beta = \sum_{\alpha \in \Pi} d_\alpha \alpha$ is any other root of Ψ , then $c_\alpha \geq d_\alpha$ for all $\alpha \in \Pi$. \square

Lemma 2.38 readily implies

$$(2.39) \quad \text{ht}(e) = \sum_{\alpha \in \Pi} m_\alpha c_\alpha.$$

The identity (2.39) is also observed in [32, §2.1].

For the remainder of this section we assume that G is simple. The generalization of each of the subsequent results to the case when G is reductive is straightforward.

For P a parabolic subgroup of G we set $\mathfrak{p}_u = \text{Lie } R_u(P)$.

Proposition 2.40. *Let $P = LR_u(P)$ be an arbitrary parabolic subgroup of G , where L is a Levi subgroup of P . Then*

$$\kappa_G(G/L) = \kappa_L(P/L) = \kappa_L(R_u(P)) = \kappa_L(\mathfrak{p}_u).$$

Proof. Thanks to Lemma 2.31, we have $\kappa_G(G/L) = \kappa_G(G *_P P/L) = \kappa_L(P/L)$.

If we write $P = R_u(P)L$, then the bijection $P/L = R_u(P)L/L \cong R_u(P)$ gives a canonical L -equivariant isomorphism $\phi : P/L \rightarrow R_u(P)$ defined by $\phi(xL) = y$, where $x = yz$ with $y \in R_u(P)$ and $z \in L$. Thus, we have $\kappa_L(P/L) = \kappa_L(R_u(P))$.

A Springer isomorphism between the unipotent variety of G and \mathcal{N} restricts to an L -equivariant isomorphism $R_u(P) \rightarrow \mathfrak{p}_u$, e.g., see [14, Cor. 1.4], so that $\kappa_L(R_u(P)) = \kappa_L(\mathfrak{p}_u)$. \square

Remarks 2.41. (i). While the first two equalities of Proposition 2.40 hold in arbitrary characteristic, the third equality requires the characteristic of the underlying field to be zero or a good prime for G ; this assumption is required for the existence of a Springer isomorphism, cf. [14, Cor. 1.4].

(ii). Lemma 4.2 in [7] states that there is a dense L -orbit on G/B if and only if there is a dense B_L -orbit on $R_u(P)$, where B_L is a Borel subgroup of L . Notice that there is a dense L -orbit on G/B if and only if there is a dense B -orbit on G/L . In other words, $\kappa_G(G/L) = 0$ if and only if $\kappa_L(R_u(P)) = 0$. Thus, Proposition 2.40 generalizes [7, Lem. 4.2].

By Proposition 2.40, the problem of determining $\kappa_L(R_u(P))$ is equivalent to the problem of determining $\kappa_G(G/L)$. In particular, a Borel subgroup of L acts on $R_u(P)$ with a dense orbit if and only if L is a spherical subgroup of G . In fact, the latter have been classified: In characteristic zero this result was proved by M. Krämer in [26] and extended to arbitrary characteristic by J. Brundan in [7, Thm. 4.1]:

Theorem 2.42. *Let L be a proper Levi subgroup of a simple group G . Then L is spherical in G if and only if (G, \mathcal{DL}) is one of $(A_n, A_{i-1}A_{n-i})$, (B_n, B_{n-1}) , (B_n, A_{n-1}) , (C_n, C_{n-1}) , (C_n, A_{n-1}) , (D_n, D_{n-1}) , (D_n, A_{n-1}) , (E_6, D_5) , or (E_7, E_6) .*

We also recall the classification of the parabolic subgroups of G with an abelian unipotent radical, cf. [41, Lem. 2.2].

Lemma 2.43. *Let G be a simple algebraic group and P be a parabolic subgroup of G . Then $R_u(P)$ is abelian if and only if P is a maximal parabolic subgroup of G which is conjugate to the standard parabolic subgroup P_I of G , where $I = \Pi \setminus \{\alpha\}$ and α occurs in the highest root $\tilde{\alpha}$ with coefficient 1.*

Let $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be a set of simple roots of the root system Ψ of G . Using Lemma 2.43, we can readily determine the standard parabolic subgroups P_I of G with an abelian unipotent radical. For G simple we gather this information in Table 1 below along with the structure of the corresponding standard Levi subgroup L_I of P_I . Set $P_{\alpha'_i} = P_{\Pi \setminus \{\alpha_i\}}$. Here the simple roots are labelled as in [3, Planches I - IX].

Note that if G is of type E_8 , F_4 or G_2 , then G does not admit a parabolic subgroup with an abelian unipotent radical. Also compare the list of pairs (G, \mathcal{DL}) from Table 1 with the list in Theorem 2.42.

Our next result is immediate from [7, Thm. 4.1, Lem. 4.2].

Proposition 2.44. *If $P = LR_u(P)$ is a parabolic subgroup of G with $R_u(P)$ abelian, then $\kappa_L(R_u(P)) = 0$.*

Proof. If $R_u(P)$ is abelian, then using Table 1 we see that all the possible pairs (G, \mathcal{DL}) appear in the list of spherical Levi subgroups given in Theorem 2.42, that is $\kappa_G(G/L) = 0$. Proposition 2.40 then implies that $\kappa_L(R_u(P)) = 0$. \square

Type of G	P_I	Type of \mathcal{DL}_I
A_n	$P_{\alpha'_i}$ for $1 \leq i \leq n$	$A_{i-1}A_{n-i}$
B_n	$P_{\alpha'_1}$	B_{n-1}
C_n	$P_{\alpha'_n}$	A_{n-1}
D_n	$P_{\alpha'_1}, P_{\alpha'_{n-1}}$ and $P_{\alpha'_n}$	D_{n-1} or A_{n-1}
E_6	$P_{\alpha'_1}$ and $P_{\alpha'_6}$	D_5
E_7	$P_{\alpha'_7}$	E_6

TABLE 1. Parabolic Subgroups with Abelian Unipotent Radical.

Corollary 2.45. *If P is a parabolic subgroup of G with $R_u(P)$ abelian, then $\kappa_L(\mathfrak{p}_u) = 0$.*

Let Ψ be the root system of G and let $\Pi \subseteq \Psi$ be a set of simple roots of Ψ . Let $P = P_I$ ($I \subseteq \Pi$) be a standard parabolic subgroup of G . Let Ψ_I be the root system of the standard Levi subgroup L_I , i.e., Ψ_I is spanned by I . Define $\Psi_I^+ = \Psi_I \cap \Psi^+$. For any root $\alpha \in \Psi$ we can uniquely write $\alpha = \alpha_I + \alpha_{I'}$ where $\alpha_I = \sum_{\beta \in I} c_\beta \beta$ and $\alpha_{I'} = \sum_{\beta \in \Pi \setminus I} d_\beta \beta$. We define the *level of α (relative to P or relative to I)* to be

$$\text{lv}(\alpha) := \sum_{\beta \in \Pi \setminus I} d_\beta,$$

cf. [1]. Let d be the maximal level of any root in Ψ . If $2i > d$, then

$$A_i := \prod_{\text{lv}(\alpha)=i} U_\alpha$$

is an abelian unipotent subgroup of G . Note A_d is the centre of $R_u(P)$. Since L normalizes each A_i , we can consider $\kappa_L(A_i)$.

Proposition 2.46. *If P is a parabolic subgroup of G and $2i > d$, then $\kappa_L(A_i) = 0$.*

Proof. We maintain the setup from the previous paragraph. Setting $A_i = \prod_{\text{lv}(\alpha)=i} U_\alpha$ and $A_i^- = \prod_{\text{lv}(\alpha)=-i} U_\alpha$, let H be the subgroup of G generated by A_i , A_i^- , and L . Then H is reductive, with root system $\Psi_I \cup \{\alpha \in \Psi \mid \text{lv}(\alpha) = \pm i\}$, and LA_i is a parabolic subgroup of H . Since A_i is abelian, we can invoke Proposition 2.44 to deduce that $\kappa_L(A_i) = 0$. \square

There is a natural Lie algebra analogue of Proposition 2.46: Maintaining the setup from above, for $2i > d$, we see that $\mathfrak{a}_i := \bigoplus_{\text{lv}(\alpha)=i} \mathfrak{g}_\alpha$ is an abelian subalgebra of \mathfrak{g} . Since $\text{Lie } U_\alpha = \mathfrak{g}_\alpha$ for all $\alpha \in \Psi$, we have $\text{Lie } A_i = \mathfrak{a}_i$. Thanks to [14, Cor. 1.4], we obtain the following consequence of Proposition 2.46.

Corollary 2.47. *If P is a parabolic subgroup of G and $2i > d$, then $\kappa_L(\mathfrak{a}_i) = 0$.*

Remarks 2.48. (i). Corollary 2.47 was first proved, for a field of characteristic zero, in [32, Prop. 3.2], although the proof there is somewhat different from ours.

(ii). Propositions 2.44 and 2.46 suggest that if A is an abelian subgroup of $R_u(P)$ which is normal in P , then $\kappa_L(A) = 0$. It is indeed the case that P acts on A with a dense orbit, see [42, Thm. 1.1]. However, this is not the case when we consider instead the action of a Borel subgroup of a Levi subgroup of P on A . For example, it follows from [42, Table

1] that if G is of type A_n , then the dimension of a maximal normal abelian subgroup A of a Borel subgroup B of G is $i(n+1-i)$, where $1 \leq i \leq n$. Clearly, for $1 \neq i \neq n$ we have $\dim A > \operatorname{rk} G$. Thus, a maximal torus of B cannot act on A with a dense orbit. Using [42, Table 1], it is easy to construct further examples.

3. THE CLASSIFICATION OF THE SPHERICAL NILPOTENT ORBITS

3.1. Height Two Nilpotent Orbits. In this subsection we show that height two nilpotent orbits are spherical. Let $e \in \mathfrak{g}$ be nilpotent and let $\lambda \in \Omega_G^a(e)$ be an associated cocharacter of G . Define the following subalgebra of \mathfrak{g} :

$$(3.1) \quad \mathfrak{g}_E := \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(2i).$$

Proposition 3.2. *Let $e \in \mathcal{N}$, $\lambda \in \Omega_G^a(e)$, and let \mathfrak{g}_E be the subalgebra of \mathfrak{g} defined in (3.1).*

- (i) *There exists a connected reductive subgroup G_E of G such that $\operatorname{Lie} G_E = \mathfrak{g}_E$.*
- (ii) *There exists a parabolic subgroup Q of G_E such that $\operatorname{Lie} Q = \bigoplus_{i \geq 0} \mathfrak{g}(2i)$. Moreover, $C_G(\lambda)$ is a Levi subgroup of Q and $\operatorname{Lie} R_u(Q) = \bigoplus_{i \geq 1} \mathfrak{g}(2i)$.*

Proof. Fix a maximal torus T of G such that $\lambda(k^*) \leq T$. Set $\Phi = \{\alpha \in \Psi \mid \langle \alpha, \lambda \rangle \in 2\mathbb{Z}\}$. Then $\mathfrak{g}_E = \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$.

Then Φ is a semisimple subsystem of Ψ . The subgroup G_E generated by T and all the one-dimensional root subgroups U_α with $\alpha \in \Phi$ is reductive and has Lie algebra \mathfrak{g}_E .

Let $Q = P \cap G_E$, where $P = P_\lambda$. Since $\lambda(k^*) \leq T \leq G_E$, we see that Q is a parabolic subgroup of G_E , see the remarks preceding Theorem 2.16. Since $\operatorname{Lie} C_G(\lambda) = \mathfrak{g}(0)$, we have $C_G(\lambda) \leq Q$ and so $C_G(\lambda)$ is a Levi subgroup of Q . The remaining claims follow from the fact that $\operatorname{Lie} P = \mathfrak{g}_{\geq 0}$, the parabolic subgroup P has Levi decomposition $P = C_G(\lambda)R_u(P)$ and $\operatorname{Lie} R_u(P) = \mathfrak{g}_{>0}$. \square

The following discussion and Lemma 3.3 allow us to reduce the determination of the spherical nilpotent orbits to the case when G is simple. Since the centre of G acts trivially on \mathfrak{g} , we may assume that G is semisimple. Let \tilde{G} be semisimple of adjoint type and $\pi : G \rightarrow \tilde{G}$ be the corresponding isogeny. Let $e \in \mathfrak{g}$ be nilpotent and let $\tilde{e} = d\pi_1(e)$. Consider the restriction of $d\pi_1$ to the nilpotent variety of \mathfrak{g} . Then $d\pi_1 : \mathcal{N} \rightarrow \tilde{\mathcal{N}}$ is a dominant G -equivariant morphism, where $\tilde{\mathcal{N}}$ is the nilpotent variety of $\operatorname{Lie} \tilde{G}$ and G acts on $\tilde{\mathcal{N}}$ via $\tilde{\alpha} \circ \pi$. It then follows from Theorem 2.8 that $\kappa_G(G \cdot e) = \kappa_{\tilde{G}}(\tilde{G} \cdot \tilde{e})$. We therefore may assume that G is semisimple of adjoint type.

Lemma 3.3. *Let G be semisimple of adjoint type. Then G is a direct product of simple groups $G = G_1 G_2 \cdots G_r$. If $e \in \mathfrak{g}$ is nilpotent, then $e = e_1 + e_2 + \cdots + e_r$ for e_i nilpotent in $\mathfrak{g}_i = \operatorname{Lie} G_i$ and $\kappa_G(G \cdot e) = \sum_{i=1}^r \kappa_{G_i}(G_i \cdot e_i)$.*

Proof. Since G is semisimple of adjoint type, so that G is the direct product $G = G_1 G_2 \cdots G_r$ of simple groups G_i , we have $\operatorname{Lie} G = \bigoplus \operatorname{Lie} G_i$. Let $e \in \mathfrak{g}$ be nilpotent. Clearly, any element $x \in C_G(e)$ is of the form $x = x_1 x_2 \cdots x_r$ where $x_i \in G_i$ and we also have that $e = e_1 + e_2 + \cdots + e_r$, where $e_i \in \mathfrak{g}_i$ and each e_i must be nilpotent. We know that $\operatorname{Ad}(x)(e) = e$ so $\operatorname{Ad}(x_1) \operatorname{Ad}(x_2) \cdots \operatorname{Ad}(x_r)(e_1 + e_2 + \cdots + e_r) = e_1 + e_2 + \cdots + e_r$. For $i \neq j$ we have $\operatorname{Ad}(x_i)(e_j) = e_j$, so $\operatorname{Ad}(x)(e_i) = \operatorname{Ad}(x_i)(e_i)$. Therefore, as $\operatorname{Ad}(x_i)$ stabilizes \mathfrak{g}_i , we have $\operatorname{Ad}(x_i)(e_i) = e_i$. Thus, we obtain the following decomposition $C_G(e) = C_{G_1}(e_1) C_{G_2}(e_2) \cdots C_{G_r}(e_r)$. For B a

Borel subgroup of G we have $B = B_1 B_2 \cdots B_r$, where each B_i is a Borel subgroup of G_i and $C_B(e) = C_{B_1}(e_1) C_{B_2}(e_2) \cdots C_{B_r}(e_r)$. In particular, for $B \in \Gamma_G(e)$ we have that $\dim C_B(e)$ is minimal. This implies that $\dim C_{B_i}(e_i)$ is minimal for each i and so $B_i \in \Gamma_{G_i}(e_i)$. Therefore, we have

$$\begin{aligned} \kappa_G(G \cdot e) &= \dim G - \dim C_G(e) - \dim B + \dim C_B(e) \\ &= \sum_{i=1}^r \dim G_i - \sum_{i=1}^r \dim C_{G_i}(e_i) - \sum_{i=1}^r \dim B_i + \sum_{i=1}^r \dim C_{B_i}(e_i) \\ &= \sum_{i=1}^r (\dim G_i - \dim C_{G_i}(e_i) - \dim B_i + \dim C_{B_i}(e_i)) \\ &= \sum_{i=1}^r \kappa_{G_i}(G_i \cdot e_i), \end{aligned}$$

and the result follows. \square

Lemma 3.4. *Let G be a connected reductive algebraic group and $e \in \mathfrak{g}$ be nilpotent. If $\text{ht}(e) = 2$, then e is spherical.*

Proof. First we assume that G is simple. Let $\lambda \in \Omega_G^a(e)$. Let \mathfrak{g}_E be the Lie subalgebra of \mathfrak{g} as defined in (3.1) and let Q be the parabolic subgroup of G_E as in Proposition 3.2(ii). Since $\text{ht}(e) = 2$, we have $\mathfrak{g}_E = \mathfrak{g}(-2) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(2)$. Set $L = C_G(\lambda)$. Then $\kappa_G(G \cdot e) = \kappa_L(\mathfrak{g}(2))$, by Theorem 2.33. Also, by Proposition 3.2, $\text{Lie } R_u(Q) = \mathfrak{g}(2)$. Since $R_u(Q)$ is abelian, Corollary 2.45 implies that $\kappa_L(\mathfrak{g}(2)) = 0$.

Now suppose that G is reductive. Let $\mathcal{D}G = G_1 G_2 \cdots G_r$ be a commuting product of simple groups. For $e \in \mathfrak{g}$ we have $e = e_1 + e_2 + \cdots + e_r$, where $e_i \in \mathfrak{g}_i = \text{Lie } G_i$ and each e_i is nilpotent. Since $\text{ht}(e) = \max_{1 \leq i \leq r} \text{ht}(e_i)$, we have $\text{ht}(e_i) \leq \text{ht}(e) = 2$ for all i . Since $\kappa_G(G \cdot e) = \sum_{i=1}^r \kappa_{G_i}(G_i \cdot e_i)$, by Lemma 3.3, the result follows from the simple case just proved. \square

3.2. Even Gradings. Suppose that the given nilpotent element $e \in \mathfrak{g}$ satisfies $\text{ht}(e) \geq 4$. Also assume that any $\lambda \in \Omega_G^a(e)$ induces an *even grading* on \mathfrak{g} , that is $\mathfrak{g}(i, \lambda) = \{0\}$ whenever i is odd. As usual we denote $\mathfrak{g}(i, \lambda)$ simply by $\mathfrak{g}(i)$.

Lemma 3.5. *Let $e \in \mathcal{N}$ and $\lambda \in \Omega_G^a(e)$ be as above. Then $\mathfrak{g}_{\geq 2}$ is non-abelian.*

Proof. Set $\text{ht}(e) = d$. For the highest root $\tilde{\alpha} \in \Psi^+$ we have $\mathfrak{g}_{\tilde{\alpha}} \subseteq \mathfrak{g}(d)$. Write $\tilde{\alpha} = \alpha_1 + \alpha_2 + \cdots + \alpha_r$ as a sum of not necessarily distinct simple roots. The sequence of simple roots $\alpha_1, \alpha_2, \dots, \alpha_r$ can be chosen so that $\alpha_1 + \alpha_2 + \cdots + \alpha_s$ is a root for all $1 \leq s \leq r$, [19, Cor. 10.2.A]. Since the grading of \mathfrak{g} induced by λ is even, for all simple roots $\alpha \in \Pi$, we have $\mathfrak{g}_\alpha \subseteq \mathfrak{g}(i)$ with $i \in \{0, 2\}$, cf. (2.37). Since $d \geq 4$, for at least one α_i we must have $\mathfrak{g}_{\alpha_i} \subseteq \mathfrak{g}(2)$. Let α_k be the last simple root in the sequence $\alpha_1, \alpha_2, \dots, \alpha_r$ with this property. Thus, for $\beta = \alpha_1 + \alpha_2 + \cdots + \alpha_{k-1}$ we have $\mathfrak{g}_\beta \in \mathfrak{g}(d-2) \subseteq \mathfrak{g}_{\geq 2}$. Since $\text{char } k$ is good for G , we have $[\mathfrak{g}_\beta, \mathfrak{g}_{\alpha_k}] = \mathfrak{g}_{\beta'}$ where $\beta' = \beta + \alpha_k$. Therefore, $\mathfrak{g}_{\geq 2}$ is non-abelian. \square

Corollary 3.6. *Let P be the destabilizing parabolic subgroup of G defined by $e \in \mathcal{N}$. Then $R_u(P)$ is non-abelian.*

Set $\mathfrak{p}_u = \text{Lie } R_u(P)$. Because the grading of \mathfrak{g} is even, $\mathfrak{g}_{\geq 2} = \mathfrak{p}_u$. Thus, by Proposition 2.40 and Theorem 2.33, we have $\kappa_G(G \cdot e) = \kappa_G(G/L)$, where $L = C_G(\lambda)$. Using the classification of the spherical Levi subgroups and the classification of the parabolic subgroups of G with abelian unipotent radical, Theorem 2.42 and Lemma 2.43, we see that there are only two cases, for G simple, when $R_u(P)$ is non-abelian and L is spherical, namely when G is of type B_n and \mathcal{DL} is of type A_{n-1} and when G is of type C_n and \mathcal{DL} is of type C_{n-1} .

Lemma 3.7. *Let G be of type B_n or of type C_n . Let $e \in \mathcal{N}$ and $\lambda \in \Omega_G^a(e)$. Set $L = C_G(\lambda)$. If $\pi_e = [1^{r_1}, 2^{r_2}, \dots]$ is the corresponding partition for e , then $\dim Z(L) = |\{a_i, b_i \in \mathbb{Z}_{\geq 0} \mid a_i + 1 = \sum_{j \geq i} r_{2j+1}, b_i + 1 = \sum_{j \geq i} r_{2j}\}|$.*

Proof. Since L is reductive, $L = Z(L)\mathcal{DL}$, and $Z(L) \cap \mathcal{DL}$ is finite, we have $\dim L = \dim Z(L) + \dim \mathcal{DL}$. The result follows from Proposition 2.36. \square

It is straightforward to deduce the following from Propositions 2.27 and 2.36.

Lemma 3.8. *Let $e \in \mathcal{N}$ and $\lambda \in \Omega_G^a(e)$ with $\text{ht}(e) \geq 4$. Set $L = C_G(\lambda)$. If G is of type B_n , then \mathcal{DL} is not of type A_{n-1} and if G is of type C_n , then \mathcal{DL} is not of type C_{n-1} .*

Lemma 3.9. *Let $e \in \mathcal{N}$ and suppose that $\lambda \in \Omega_G^a(e)$ induces an even grading on \mathfrak{g} . If $\text{ht}(e) \geq 4$, then e is non-spherical.*

Proof. First we observe that if G is simple, then the statement follows from the facts that $R_u(P)$ is non-abelian (Corollary 3.6) and that (G, \mathcal{DL}) is not one of the pairs (B_n, A_{n-1}) or (C_n, C_{n-1}) (Lemma 3.8). So by Theorem 2.42 and Lemma 2.43, we see that L is a non-spherical subgroup. Therefore, by Proposition 2.40, $\kappa_L(\mathfrak{g}_{\geq 2}) > 0$ and e is non-spherical.

In case G is reductive, we argue as in the proof of Lemma 3.4 and reduce to the simple case. \square

3.3. Nilpotent Orbits of Height at Least Four. Let $e \in \mathfrak{g}$ be nilpotent and let $\lambda \in \Omega_G^a(e)$. Let \mathfrak{g}_E be the subalgebra of \mathfrak{g} as defined in (3.1). Also let G_E be the connected reductive algebraic group such that $\text{Lie } G_E = \mathfrak{g}_E$ and Q be the parabolic subgroup of G_E as in Proposition 3.2(ii).

Since $e \in \mathfrak{g}_E$ and $\lambda(k^*) \leq G_E$, it follows from [11, Thm. 1.1] that λ is a cocharacter of G_E which is associated to e , i.e. $\lambda \in \Omega_{G_E}^a(e)$. Moreover, for $P = P_\lambda$, we have $Q = P \cap G_E$ is the destabilizing parabolic subgroup of G_E defined by e .

Let $\text{ht}_E(e)$ denote the height of $e \in \mathfrak{g}_E$. Now if $\text{ht}(e) \geq 4$ and $\text{ht}(e)$ is even, then $\text{ht}_E(e) = \text{ht}(e)$. The case when $\text{ht}(e) \geq 4$ and $\text{ht}(e)$ is odd is slightly more involved. First we need some preliminary results. A proof of the following can be found in [33, Prop. 2.4].

Lemma 3.10. *Suppose that $\text{char } k = 0$. If $e \in \mathcal{N}$ with $\text{ht}(e)$ odd, then the weighted Dynkin diagram $\Delta(e)$ contains no “2” labels.*

If Π is a set of simple roots of Ψ relative to a maximal torus T which contains $\lambda(k^*)$, then for $\alpha \in \Pi$ we have

$$(3.11) \quad \mathfrak{g}_\alpha \subseteq \mathfrak{g}(i) \text{ where } i \in \{0, 1\}.$$

To see this recall (2.37): $\text{Ad}(\lambda(t))(e_\alpha) = t^{m_\alpha} e_\alpha$, for $e_\alpha \in \mathfrak{g}_\alpha$ and m_α is the corresponding label of the weighted Dynkin diagram $\Delta(e)$ of e . Thus, by Lemma 3.10, we have $m_\alpha \in \{0, 1\}$.

Lemma 3.12. *If $\text{ht}(e) = d$ odd, then $\mathfrak{g}(d-1) \neq \{0\}$.*

Proof. The result follows easily, arguing as in the proof of Lemma 3.5 and using (3.11). \square

Corollary 3.13. *If $e \in \mathcal{N}$ with $\text{ht}(e)$ odd, then $\text{ht}_E(e) = \text{ht}(e) - 1$.*

In particular, we have the following conclusion.

Corollary 3.14. *If $e \in \mathcal{N}$ with $\text{ht}(e) \geq 4$, then $\text{ht}_E(e) \geq 4$.*

Thus, by Lemma 3.9, Corollary 3.14, and the fact that $\Omega_G^a(e) \cap Y(G_E) = \Omega_{G_E}^a(e)$ ([11, Thm. 1.1]), we have $\kappa_L(\mathfrak{g}_{E, \geq 2}) > 0$, where $\mathfrak{g}_{E, \geq 2} = \bigoplus_{i \geq 1} \mathfrak{g}(2i)$ and $L = C_G(\lambda) = C_{G_E}(\lambda)$.

Lemma 3.15. *If a Borel subgroup B_L of L acts on $\mathfrak{g}_{\geq 2}$ with a dense orbit, then B_L acts on $\mathfrak{g}_{E, \geq 2}$ with a dense orbit.*

Proof. This follows readily from Theorem 2.9. \square

Combining Lemmas 3.9, 3.15 and Corollary 3.14, we get the main result of this subsection.

Proposition 3.16. *Let $e \in \mathcal{N}$. If $\text{ht}(e) \geq 4$, then e is non-spherical.*

3.4. Nilpotent Orbits of Height Three. Let $e \in \mathcal{N}$ and let $\lambda \in \Omega_G^a(e)$. Let $P = P(e)$ be the destabilizing parabolic subgroup defined by e . Then $P = LR_u(P)$ for $L = C_G(\lambda)$. Let B_L be a Borel subgroup of L so that $\lambda(k^*) \leq B_L$. Write $B_L = TU_L$ for a Levi decomposition of B_L , where $U_L = R_u(B_L)$ and T is a maximal torus of G containing $\lambda(k^*)$. Let $\mathfrak{b}_L = \text{Lie } B_L$, $\mathfrak{n} = \text{Lie } U_L$, and $\mathfrak{t} = \text{Lie } T$.

Lemma 3.17. *Let $e \in \mathfrak{g}$ be nilpotent and λ be an associated cocharacter for e in \mathfrak{g} . Then the following are equivalent.*

- (i) *The nilpotent element e is spherical.*
- (ii) *There exists $e' \in \mathfrak{g}_{\geq 2}$ such that $\overline{\text{Ad}(B_L)(e')} = \mathfrak{g}_{\geq 2}$.*
- (iii) *There exists $e' \in \mathfrak{g}_{\geq 2}$ such that $\dim C_{B_L}(e') = \dim B_L - \dim \mathfrak{g}_{\geq 2}$.*

Proof. Thanks to Theorem 2.33, $\kappa_G(G \cdot e) = \kappa_L(\mathfrak{g}_{\geq 2})$. Thus (i) and (ii) are equivalent. The equivalence between (ii) and (iii) is clear. \square

Recall from Subsection 2.1 the definition of the support of a nilpotent element in \mathfrak{u} .

Lemma 3.18. *Let $e \in \mathfrak{g}_{\geq 2}$. If $\text{supp}(e)$ is linearly independent, then $\dim C_T(e) = \dim T - |\text{supp}(e)|$.*

Proof. Suppose that $\text{supp}(e)$ is linearly independent. Then $\dim \text{Ad}(T)(e) = |\text{supp}(e)|$, e.g. see [13, Lem. 3.2]. The desired equality follows. \square

The following is a standard consequence of orbit maps.

Lemma 3.19. *Let $e' \in \mathfrak{g}_{\geq 2}$. Then $\dim C_{B_L}(e') \leq \dim \mathfrak{b}_L(e')$ and $\dim C_{U_L}(e') \leq \dim \mathfrak{n}(e')$.*

In [15, Prop. 5.4], Goodwin showed that each U -orbit in \mathfrak{u} admits a unique so called *minimal* orbit representative, see [15, Def. 5.3]. (This depends on a suitable choice of an ordering of the positive roots compatible with the height function, cf. [15, Def. 3.1].) Moreover, a special case of [15, Prop. 7.7] gives that for e the minimal representative of its U -orbit in \mathfrak{u} , we have $C_B(e) = C_T(e)C_U(e)$. As a consequence, we readily obtain the following.

Lemma 3.20. *Let $e' \in \mathfrak{g}_{\geq 2}$. Suppose that e' is the minimal representative of its U -orbit in \mathfrak{u} . Then $C_{B_L}(e') = C_T(e')C_{U_L}(e')$. In particular, $\dim C_{B_L}(e') = \dim C_T(e') + \dim C_{U_L}(e')$.*

Proposition 3.21. *Let G be a simple algebraic group. Table 2 below gives a complete list of the height 3 nilpotent orbits in \mathfrak{g} .*

Proof. For the classical groups we use Proposition 2.27. By Remark 2.28, there are no height 3 nilpotent orbits in types A_n and C_n . Using the tables in [10, §13] and (2.39), one readily determines the desired orbits when G is exceptional. \square

In Table 2 we either give the partition or the Bala–Carter label of the corresponding orbit, cf. [10, §13].

Type of G	Orbits
A_n	-
B_n	$[1^j, 2^{2i}, 3]$ with $i > 0$
C_n	-
D_n	$[1^j, 2^{2i}, 3]$ with $i > 0$
G_2	\tilde{A}_1
F_4	$A_1 + \tilde{A}_1$
E_6	$3A_1$
E_7	$(3A_1)', 4A_1$
E_8	$3A_1, 4A_1$

TABLE 2. The nilpotent orbits of height 3.

In the next three subsections we concentrate on the height 3 orbits in types B_n , D_n , and the exceptional types, respectively.

3.5. Height Three Nilpotent Elements of $\mathfrak{so}_{2n+1}(k)$. In this subsection let G be of type B_n for $n \geq 3$, so $\mathfrak{g} = \mathfrak{so}_{2n+1}(k)$. The nilpotent orbits in \mathfrak{g} are classified by the partitions of $2n + 1$ with even parts occurring with even multiplicity, see [22, Thm. 1.6]. By Proposition 2.27, the height 3 nilpotent orbits correspond to partitions of $2n + 1$ of the form $\pi_{r,s} = [1^s, 2^{2r}, 3]$, where $r \geq 1, s \geq 0$ and $2r + s + 1 = n$. Denote the corresponding nilpotent orbit by $\mathcal{O}_{r,s}$ and a representative of such an orbit by $e_{r,s}$.

Lemma 3.22. *There are precisely $\lfloor \frac{n-1}{2} \rfloor$ distinct height 3 nilpotent orbits in \mathfrak{g} .*

Proof. By our comments above, we need to show that there are precisely $\lfloor \frac{n-1}{2} \rfloor$ partitions of $2n + 1$ of the form $\pi_{r,s}$. This is equivalent to finding all partitions of $n - 1$ of the form $[1^{s/2}, 2^r]$. Thus r satisfies $1 \leq r \leq \frac{n-1}{2}$. Since r is an integer, the result follows. \square

Since the number $2r + 1$ appears frequently in the sequel, we set $\hat{r} = 2r + 1$. Using [10, §13], we readily see that that $e_{r,s}$ has the following weighted Dynkin diagram:

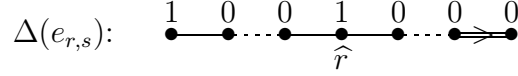


FIGURE 1. Labeling of $\Delta(e_{r,s})$.

Remark 3.23. Note that in $\Delta(e_{r,s})$ there are precisely two simple roots, α_1 and $\alpha_{\widehat{r}}$ that are labeled with a “1” and that there is an odd number of simple roots between α_1 and $\alpha_{\widehat{r}}$. Also, the short simple root is labeled with a “1” if and only if $s = 0$, and this can only happen when n is odd.

We refer to [3, Planche II] for information regarding the root system of type B_n . Let $\alpha_1, \dots, \alpha_n$ be the simple roots of Ψ^+ and let

$$\begin{aligned}\beta_{j,k} &= \alpha_j + \dots + \alpha_k \text{ for } 1 \leq j \leq k \leq n, \\ \gamma_{j,k} &= \alpha_j + \dots + \alpha_{k-1} + 2\alpha_k + \dots + 2\alpha_n \text{ for } 1 \leq j < k < n,\end{aligned}$$

where $\beta_{j,j} = \alpha_j$. Note that all the possible β 's and γ 's exhaust Ψ^+ .

For a T -stable Lie subalgebra \mathfrak{m} of \mathfrak{u} recall the definition of the set of roots $\Psi(\mathfrak{m})$ of \mathfrak{m} with respect to T from Subsection 2.1.

Lemma 3.24. *For an associated cocharacter of $e_{r,s}$ in \mathfrak{g} we have*

- (i) $\Psi(\mathfrak{g}(2)) = \{\beta_{1,j}, \gamma_{i,m}, \gamma_{l,k} \mid 1 < l < k \leq \widehat{r} \leq j \text{ and } 1 < i < m \leq \widehat{r}\}$, and so $\dim \mathfrak{g}(2) = 2r^2 - r + 2s + 1$;
- (ii) $\Psi(\mathfrak{g}(3)) = \{\gamma_{1,k} \mid k \leq \widehat{r}\}$, and so $\dim \mathfrak{g}(3) = 2r$.

Proof. For every $\delta \in \Psi$ we have that $\mathfrak{g}_\delta \subseteq \mathfrak{g}(i)$ for some $i \in \{0, \pm 1, \pm 2, \pm 3\}$. For the simple roots this information can be read off from $\Delta(e_{r,s})$, see (2.37). Let $\delta = \sum_{\alpha \in \Pi} c_{\delta,\alpha} \alpha$ be a positive root.

Now $\mathfrak{g}_\delta \subseteq \mathfrak{g}(2)$ if and only if $c_{\delta,\alpha_1} + c_{\delta,\alpha_{\widehat{r}}} = 2$. All of the roots listed above satisfy this condition, and no others do. Finally, $\mathfrak{g}_\delta \subseteq \mathfrak{g}(3)$ if and only if $c_{\delta,\alpha_1} + c_{\delta,\alpha_{\widehat{r}}} = 3$. All of the roots listed above satisfy this condition, and no others do. \square

Lemma 3.25. *For an associated cocharacter of $e_{r,s}$ in \mathfrak{g} we have*

- (i) $\Psi(\mathfrak{b}_L) = \{\beta_{j,k}, \gamma_{l,m} \mid \widehat{r} < j \text{ or } 1 < j \leq k < \widehat{r}, \widehat{r} < l < m\}$.
- (ii) $\dim \mathfrak{b}_L = 2r^2 + s^2 + s + r + 1$.

Proof. For every $\delta \in \Psi$ we have that $\mathfrak{g}_\delta \subseteq \mathfrak{g}(i)$ for some $i \in \{0, \pm 1, \pm 2, \pm 3\}$. As mentioned above, for the simple roots this information can be read off from $\Delta(e_{r,s})$, see (2.37). Let $\delta = \sum_{\alpha \in \Pi} c_{\delta,\alpha} \alpha \in \Psi^+$. Then $\mathfrak{g}_\delta \subseteq \mathfrak{b}_L$ if and only if $c_{\delta,\alpha_1} + c_{\delta,\alpha_{\widehat{r}}} = 0$. All of the roots listed above satisfy this condition, and no others do. Consequently, $\dim \mathfrak{n} = 2r^2 + s^2 - r$. Since $\dim \mathfrak{t} = n$, we get $\dim \mathfrak{b}_L = 2r^2 + s^2 + s + r + 1$. \square

It follows from Figure 1 that L is of Dynkin type $A_{\widehat{r}-1} \times B_s$. Accordingly, there is a natural partition of the roots of \mathfrak{b}_L into a union of two subsets, namely the positive roots of the $A_{\widehat{r}-1}$ and B_s subsystems, respectively. Thus, we have $\Psi(\mathfrak{b}_L) = \Psi_1(\mathfrak{b}_L) \cup \Psi_2(\mathfrak{b}_L)$, where

$$\begin{aligned}\Psi_1(\mathfrak{b}_L) &= \{\beta_{j,k} \mid 1 < j \leq k < \widehat{r}\}, \\ \Psi_2(\mathfrak{b}_L) &= \{\beta_{j,k}, \gamma_{l,m} \mid \widehat{r} < j \leq k, \widehat{r} < l < m\}.\end{aligned}$$

Similarly, we can decompose the roots of $\mathfrak{g}_{\geq 2}$ into two sets as follows: $\Psi(\mathfrak{g}_{\geq 2}) = \Psi_1(\mathfrak{g}_{\geq 2}) \cup \Psi_2(\mathfrak{g}_{\geq 2})$, where

$$\begin{aligned}\Psi_1(\mathfrak{g}_{\geq 2}) &= \{\gamma_{j,k} \mid 1 \leq j < k \leq \widehat{r}\}, \\ \Psi_2(\mathfrak{g}_{\geq 2}) &= \{\beta_{1,j}, \gamma_{1,k} \mid \widehat{r} \leq j, \widehat{r} < k\}.\end{aligned}$$

The sets $\Psi_i(\mathfrak{b}_L)$ and $\Psi_i(\mathfrak{g}_{\geq 2})$ satisfy the following property:

$$(3.26) \quad \delta \in \Psi_i(\mathfrak{b}_L), \eta \in \Psi_{3-i}(\mathfrak{g}_{\geq 2}) \Rightarrow \delta + \eta \notin \Psi, \quad i \in \{1, 2\}.$$

Denote by \mathfrak{b}_L^i the Lie subalgebras of \mathfrak{b}_L such that $\Psi(\mathfrak{b}_L^i) = \Psi_i(\mathfrak{b}_L)$ for $i = 1, 2$. For the rest of this subsection we show that the following element is a representative of the dense B_L -orbit in $\mathfrak{g}_{\geq 2}$; set:

$$e'_{r,s} := \sum_{j,k=0}^{r-1} (e_{\gamma_{\widehat{r}-2j-1, \widehat{r}-2j}} + e_{\gamma_{1, \widehat{r}-2k}}) + e_{\gamma_{1, \widehat{r}+1}} + e_{\beta_{1, \widehat{r}}},$$

where $e_\delta \in \mathfrak{g}_\delta \setminus \{0\}$ for $\delta \in \Psi(\mathfrak{g}_{\geq 2})$.

Recall from the paragraph before Lemma 3.20 the notion of minimal U -orbit representatives in \mathfrak{u} from [15].

Lemma 3.27. *Each $e'_{r,s}$ is the minimal representative of its U -orbit in \mathfrak{u} , $\text{supp}(e'_{r,s})$ is linearly independent, and $|\text{supp}(e'_{r,s})| = \begin{cases} 2r+2 & \text{if } s > 0; \\ 2r+1 & \text{if } s = 0. \end{cases}$*

Proof. It is straightforward to check that $e'_{r,s}$ is the minimal representative of its U -orbit in \mathfrak{u} in the sense of [15] and one easily computes $|\text{supp}(e'_{r,s})|$. Note that the root $\gamma_{1, \widehat{r}+1}$ only occurs if $s > 0$.

Suppose there exist scalars τ_j, ξ_k, μ and ν such that

$$\sum_{j=0}^{r-1} \tau_j \gamma_{\widehat{r}-2j-1, \widehat{r}-2j} + \sum_{k=0}^{r-1} \xi_k \gamma_{1, \widehat{r}-2k} + \mu \gamma_{1, \widehat{r}+1} + \nu \beta_{1, \widehat{r}} = 0.$$

Since the coefficients of α_1, α_2 , and α_3 must be zero, we have

$$\sum_{k=0}^{r-1} \xi_k + \mu + \nu = 0, \quad \tau_{r-1} + \sum_{k=0}^{r-1} \xi_k + \mu + \nu = 0, \quad \text{and} \quad \xi_{r-1} + 2\tau_{r-1} + \sum_{k=0}^{r-1} \xi_k + \mu + \nu = 0.$$

These three equations imply that $\tau_{r-1} = 0 = \xi_{r-1}$. Continuing in this way, we see that $\tau_j = 0 = \xi_j$ for all j . Thus we are left to show that $\gamma_{1, \widehat{r}+1}$ and $\beta_{1, \widehat{r}}$ are linearly independent; but this is obvious. \square

Thanks to Lemma 3.27 it is harmless to assume that $\text{supp}(e'_{r,s})$ is part of a Chevalley basis of \mathfrak{g} .

Lemma 3.28. $\dim \mathfrak{c}_n(e'_{r,s}) = \begin{cases} (s-1)^2 & \text{if } s > 0; \\ 0 & \text{if } s = 0. \end{cases}$

Proof. Thanks to (3.26), we may consider the two summands $\sum_{j,k=0}^{r-1} (e_{\gamma_{\widehat{r}-2j-1, \widehat{r}-2j}} + e_{\gamma_{1, \widehat{r}-2k}})$ and $e_{\gamma_{1, \widehat{r}+1}} + e_{\beta_{1, \widehat{r}}}$ of $e'_{r,s}$ separately. Since $\gamma_{\widehat{r}-2j-1, \widehat{r}-2j} + \gamma_{1, \widehat{r}-2k} \in \Psi_1(\mathfrak{g}_{\geq 2})$, we need only consider the root spaces \mathfrak{g}_δ for $\delta \in \Psi_1(\mathfrak{b}_L)$. So let $\beta_{i,m} \in \Psi_1(\mathfrak{b}_L)$. If $m = \widehat{r} - 2l$ for some $0 \leq l < r$, then, by the Chevalley commutator relations, $[e_{\gamma_{\widehat{r}-2l+1, \widehat{r}-2(l-1)}}, \mathfrak{g}_{\beta_{i, \widehat{r}-2l}}] =$

$\mathfrak{g}_{\gamma_{i,\hat{r}-2(l-1)}}$, since $\text{char } k$ is good for G . If $m = \hat{r} - 2l - 1$ for some $0 \leq l < r$, then $[e_{\gamma_{1,\hat{r}-2l}}, \mathfrak{g}_{\beta_{i,\hat{r}-2l-1}}] = \mathfrak{g}_{\gamma_{1,i}}$. Next we observe that all the β 's above exhaust the set $\Psi_1(\mathfrak{b}_L)$. Consequently, $\mathfrak{c}_{\mathfrak{b}_L^1}(\sum_{j,k=0}^{r-1} (e_{\gamma_{\hat{r}-2j-1,\hat{r}-2j}} + e_{\gamma_{1,\hat{r}-2k}})) = \{0\}$.

Next we consider the summand $e_{\gamma_{1,\hat{r}+1}} + e_{\beta_{1,\hat{r}}}$. First observe that $[\mathfrak{n}, e_{\gamma_{1,\hat{r}+1}}] = \{0\}$, so $\mathfrak{c}_{\mathfrak{n}}(e_{\gamma_{1,\hat{r}+1}}) = \mathfrak{n}$. Secondly, the root $\beta_{1,\hat{r}}$ lies in $\Psi_2(\mathfrak{g}_{\geq 2})$. Thanks to property (3.26), we need only consider roots $\delta \in \Psi_2(\mathfrak{b}_L)$. We see that the only roots $\delta \in \Psi_2(\mathfrak{b}_L)$ with $\delta + \beta_{1,\hat{r}} \in \Psi(\mathfrak{g}_{\geq 2})$ are of the form $\beta_{\hat{r}+1,j}$ or $\gamma_{\hat{r}+1,k}$ where $\hat{r}+1 \leq j \leq n$ and $\hat{r}+1 < k \leq n$. Again the Chevalley commutator relations imply $[\mathfrak{g}_{\beta_{\hat{r}+1,j}}, e_{\beta_{1,\hat{r}}}] = \mathfrak{g}_{\beta_{1,j}}$ and $[\mathfrak{g}_{\gamma_{\hat{r}+1,k}}, e_{\beta_{1,\hat{r}}}] = \mathfrak{g}_{\gamma_{1,k}}$. We also observe that $\beta_{j,k}$ and $\gamma_{l,m}$ for $\hat{r}+1 < j, l$ have the property that $\beta_{1,\hat{r}+1} + \gamma_{l,m}, \beta_{1,\hat{r}+1} + \beta_{j,k} \notin \Psi_2(\mathfrak{g}_{\geq 2})$. All the roots above exhaust $\Psi_2(\mathfrak{b}_L)$, so we conclude that all the roots $\beta_{j,k}$ and $\gamma_{l,m}$ for $\hat{r}+1 < j, l$ of $\Psi_2(\mathfrak{b}_L)$ are all contained in $\Psi(\mathfrak{c}_{\mathfrak{n}}(e_{\beta_{1,\hat{r}}}))$. If $s > 0$, these roots form the set of positive roots of a root system of type B_{s-1} , there are exactly $(s-1)^2$ positive roots in a root system of type B_{s-1} and so $|\Psi(\mathfrak{c}_{\mathfrak{n}}(e_{\beta_{1,\hat{r}}}))| = (s-1)^2$. Therefore, $\dim \mathfrak{c}_{\mathfrak{n}}(e'_{r,s}) = (s-1)^2$, clearly, if $s = 0$ then, $\dim \mathfrak{c}_{\mathfrak{n}}(e'_{r,s}) = 0$. \square

Proposition 3.29. *The B_L -orbit of $e'_{r,s}$ is dense in $\mathfrak{g}_{\geq 2}$.*

Proof. Thanks to Lemma 3.17, it is sufficient to show that $\dim B_L = \dim C_{B_L}(e'_{r,s}) + \dim \mathfrak{g}_{\geq 2}$. Lemma 3.24 implies that $\dim \mathfrak{g}_{\geq 2} = 2r^2 + 2s + r + 1$ and Lemma 3.25 implies that $\dim B_L = 2r^2 + s^2 + s + r + 1$. By Lemma 3.27, $e'_{r,s}$ is the minimal representative of its U -orbit in \mathfrak{u} . Thus, by Lemma 3.20, we have $\dim C_{B_L}(e'_{r,s}) = \dim C_T(e'_{r,s}) + \dim C_U(e'_{r,s})$. Consequently, Lemmas 3.19, 3.27, and 3.28 imply that, for $s > 0$, $\dim C_{B_L}(e'_{r,s}) \leq n - 2r - 2 + (s-1)^2 = s^2 - s$. So

$$\dim C_{B_L}(e'_{r,s}) + \dim \mathfrak{g}_{\geq 2} \leq s^2 - s + 2r^2 + r + 2s + 1 = \dim B_L.$$

This clearly implies $\dim B_L = \dim C_{B_L}(e'_{r,s}) + \dim \mathfrak{g}_{\geq 2}$. Similarly, if $s = 0$, we get $\dim B_L = \dim C_{B_L}(e'_{r,s}) + \dim \mathfrak{g}_{\geq 2}$. \square

Corollary 3.30. $\dim C_{B_L}(e'_{r,s}) = s(s-1)$.

Finally, from Lemma 3.17 we obtain

Corollary 3.31. *If G is of type B_n and $e \in \mathcal{N}$ with $\text{ht}(e) = 3$, then e is spherical.*

3.6. Height Three Nilpotent Elements of $\mathfrak{so}_{2n}(k)$. Assume for this subsection that G is of type D_n for $n \geq 4$, so $\mathfrak{g} = \mathfrak{so}_{2n}$. We know that the nilpotent orbits in \mathfrak{g} are classified by the partitions of $2n$ with even parts occurring with even parity, see [22, Thm. 1.6]. We showed that the height three nilpotent orbits correspond to partitions of $2n$ of the form $\pi_{r,s} = [1^{2s+1}, 2^{2r}, 3]$ where $r \geq 1, s \geq 0$ and $2r + s + 2 = n$, see Proposition 2.27. Similarly to the B_n case, we denote the corresponding orbit by $\mathcal{O}_{r,s}$ and a representative of such an orbit by $e_{r,s}$. Because the proofs of the results in this subsection are virtually identical to the ones in Subsection 3.5, they are omitted.

Lemma 3.32. *There are precisely $\lfloor \frac{n-2}{2} \rfloor$ distinct height 3 nilpotent orbits in \mathfrak{g} .*

Using [10, §13], we can easily calculate that for $s > 0$, $e_{r,s}$ has the weighted Dynkin diagram $\Delta(e_{r,s})$ as shown in Figure 2 below.

Similarly, when $s = 0$, the labelling of $\Delta(e_{r,0})$ is shown in Figure 3 below.

Remark 3.33. Note that there is always an odd number of “0” labels between the first and second “1” labels in $\Delta(e_{r,s})$. If $s > 0$, then there are $s+1$ “0” labels to the right of the second “1” label. Finally, $s = 0$ only if n is even.

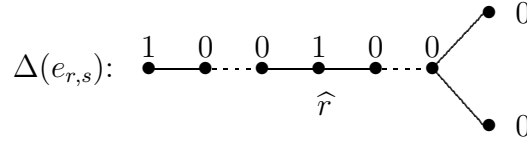


FIGURE 2. Labelling of $\Delta(e_{r,s})$ for $s > 0$.

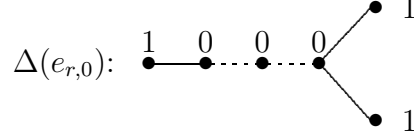


FIGURE 3. Labelling of $\Delta(e_{r,0})$.

We refer to [3, Planche IV] for information regarding the root system of type D_n . We use the following notation for the positive roots Ψ^+ . Let $\alpha_1, \dots, \alpha_n$ be the set of simple roots of Ψ^+ and let

$$\begin{aligned} \beta_{j,k} &= \alpha_j + \dots + \alpha_k \text{ for } 1 \leq j \leq k \leq n \text{ not } j = n-1, k = n, \\ \beta_j &= \alpha_j + \dots + \alpha_{n-2} + \alpha_n \text{ for } 1 \leq j \leq n-2, \\ \gamma_{j,k} &= \alpha_j + \dots + \alpha_{k-1} + 2\alpha_k + \dots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n \text{ for } 1 \leq j < k < n-2. \end{aligned}$$

Here we again use the convention $\beta_{j,j} = \alpha_j$. Note that all the possible β 's and γ 's exhaust Ψ^+ .

Next we consider the structure of the abelian Lie subalgebra $\mathfrak{g}_{\geq 2} = \mathfrak{g}(2) \oplus \mathfrak{g}(3)$.

Lemma 3.34. *An associated cocharacter for $e_{r,s}$ affords the following.*

$$(i) \quad \Psi(\mathfrak{g}(2)) = \begin{cases} \{\beta_{1,j}, \beta_1, \gamma_{l,k}, \gamma_{1,m} \mid 1 < l < k \leq \widehat{r} \leq j, \widehat{r} < m\} & \text{if } s > 0; \\ \{\beta_{1,n-1}, \beta_1, \beta_{i,n}, \gamma_{j,k} \mid 2 \leq i < \widehat{r}, 1 < j < k < \widehat{r}\} & \text{if } s = 0. \end{cases}$$

$$\text{In particular, } \dim \mathfrak{g}(2) = 2r^2 - r + 2s + 2.$$

$$(ii) \quad \Psi(\mathfrak{g}(3)) = \begin{cases} \{\gamma_{1,k} \mid k \leq \widehat{r}\} & \text{if } s > 0; \\ \{\beta_{1,n}, \gamma_{1,k} \mid 2 \leq k < \widehat{r}\} & \text{if } s = 0. \end{cases}$$

$$\text{In particular, } \dim \mathfrak{g}(3) = 2r.$$

Next we look at the structure of the Lie subalgebra \mathfrak{b}_L of $\mathfrak{g}(0)$.

Lemma 3.35. *An associated cocharacter for $e_{r,s}$ affords the following.*

$$\Psi(\mathfrak{b}_L) = \begin{cases} \{\beta_i, \beta_{j,k}, \gamma_{l,m} \mid \widehat{r} < j \text{ or } 1 < j \leq k < \widehat{r}, \widehat{r} < i, \widehat{r} < l < m\} & \text{if } s > 0; \\ \{\beta_{j,k} \mid 1 < j \leq k < \widehat{r}\} & \text{if } s = 0. \end{cases}$$

$$\text{In particular, } \dim \mathfrak{b}_L = 2r^2 + s^2 + r + 2s + 2.$$

Similarly to the B_n case, the roots of \mathfrak{b}_L naturally form two distinct subsets, namely the roots whose support lies strictly to the left of the second “1” label of the weighted Dynkin

diagram and those whose support lies strictly to the right of the second “1” label on the weighted Dynkin diagram. More precisely, we have $\Psi(\mathbf{b}_L) = \Psi_1(\mathbf{b}_L) \cup \Psi_2(\mathbf{b}_L)$ where

$$\begin{aligned}\Psi_1(\mathbf{b}_L) &= \{\beta_{j,k} \mid 1 < j \leq k < \widehat{r}\}, \\ \Psi_2(\mathbf{b}_L) &= \{\beta_{j,k}, \beta_i, \gamma_{l,m} \mid \widehat{r} < j \leq k, \widehat{r} < i, \widehat{r} < l < m\}.\end{aligned}$$

Again we partition the roots of $\mathfrak{g}_{\geq 2}$ into two distinct subsets. More precisely, we write $\Psi(\mathfrak{g}_{\geq 2}) = \Psi_1(\mathfrak{g}_{\geq 2}) \cup \Psi_2(\mathfrak{g}_{\geq 2})$, where for $s \geq 1$, we define

$$\begin{aligned}\Psi_1(\mathfrak{g}_{\geq 2}) &= \{\gamma_{j,k} \mid 1 \leq j < k \leq \widehat{r}\}, \\ \Psi_2(\mathfrak{g}_{\geq 2}) &= \{\beta_1, \beta_{1,j}, \gamma_{1,k} \mid \widehat{r} \leq j, \widehat{r} < k\},\end{aligned}$$

and for $s = 0$, we define

$$\begin{aligned}\Psi_1(\mathfrak{g}_{\geq 2}) &= \{\gamma_{j,k} \mid 1 \leq j < k \leq \widehat{r}\}, \\ \Psi_2(\mathfrak{g}_{\geq 2}) &= \{\beta_1, \beta_{1,n-1}, \beta_{j,n} \gamma_{1,k} \mid j \leq \widehat{r} < k\}.\end{aligned}$$

Again, we have the following property of these sets:

$$(3.36) \quad \delta \in \Psi_i(\mathbf{b}_L), \eta \in \Psi_{3-i}(\mathfrak{g}_{\geq 2}) \Rightarrow \delta + \eta \notin \Psi, \quad i \in \{1, 2\}.$$

For $s > 1$, set

$$e'_{r,s} := \sum_{j,k=0}^{r-1} (e_{\gamma_{\widehat{r}-2j-1, \widehat{r}-2j}} + e_{\gamma_{1, \widehat{r}-2k}}) + e_{\gamma_{1, \widehat{r}+1}} + e_{\beta_{1, \widehat{r}}} \in \mathfrak{g}_{\geq 2},$$

for $s = 1$, set

$$e'_{r,1} := \sum_{j,k=0}^{r-1} (e_{\gamma_{\widehat{r}-2j-1, \widehat{r}-2j}} + e_{\gamma_{1, \widehat{r}-2k}}) + e_{\beta_{1,n}} + e_{\beta_{1, \widehat{r}}} \in \mathfrak{g}_{\geq 2},$$

and for $s = 0$, set

$$e'_{r,0} := \sum_{j,k=1}^{r-1} (e_{\gamma_{\widehat{r}-2j-1, \widehat{r}-2j}} + e_{\gamma_{1, \widehat{r}-2k}}) + e_{\beta_{1,n}} + e_{\beta_{1,n-1}} + e_{\beta_{n-2,n}} + e_{\beta_1} \in \mathfrak{g}_{\geq 2}.$$

Lemma 3.37. *With the notation as above, we have $|\text{supp}(e'_{r,s})| = 2r + 2$, $\text{supp}(e'_{r,s})$ is linearly independent, and $\dim \mathfrak{c}_n(e'_{r,s}) = s(s-1)$.*

Proposition 3.38. *The B_L -orbit of $e'_{r,s}$ is dense in $\mathfrak{g}_{\geq 2}$.*

Corollary 3.39. *If G is of type D_n and $e \in \mathcal{N}$ with $\text{ht}(e) = 3$, then e is spherical.*

3.7. Height Three Nilpotent Elements of the Exceptional Lie Algebras. We fix an ordering of the roots $\alpha_1, \dots, \alpha_r$ of $\Psi(\mathfrak{g}_{\geq 2})$ such that $\alpha_i \prec \alpha_j$ for $i < j$. Define the subalgebra \mathfrak{m}_i of $\mathfrak{g}_{\geq 2}$ by setting $\mathfrak{m}_i = \bigoplus_{j=i+1}^r \mathfrak{g}_{\alpha_j}$ and the quotient \mathfrak{q}_i by $\mathfrak{q}_i = \mathfrak{g}_{\geq 2} / \mathfrak{m}_i$ for $0 \leq i \leq r$. Let B be a Borel subgroup of G such that $\mathfrak{g}_{\geq 2} \subseteq \text{Lie } R_u(B) = \mathfrak{u}$. Note that each \mathfrak{q}_i is a B -module.

The computer programme, DOOBS, devised by S.M. Goodwin allows us to determine whether B acts on $\mathfrak{g}_{\geq 2}$ with a dense orbit. For details of the GAP4 ([12]) computer algebra program of Goodwin, we refer the reader to [13] and [16]. Working inductively, starting with $i = 0$, at each stage of the algorithm, DOOBS determines a representative $x_i + \mathfrak{m}_i$, with

$\text{supp}(x_i)$ linearly independent of a dense B -orbit on \mathfrak{q}_i or decides that B does not act on \mathfrak{q}_i with a dense orbit.

DOOBS also keeps a record of the primes for which $\dim_p \mathfrak{c}_u(x_i + \mathfrak{m}_{i+1}) > \dim_0 \mathfrak{c}_u(x_i + \mathfrak{m}_{i+1})$, where $\dim_p \mathfrak{c}_u(x_i + \mathfrak{m}_{i+1})$ and $\dim_0 \mathfrak{c}_u(x_i + \mathfrak{m}_{i+1})$ denote the dimension of $\mathfrak{c}_u(x_i + \mathfrak{m}_{i+1})$ over a field of characteristic p and characteristic 0 respectively, see Remark 3.2 in [13]. For these primes we cannot conclude that B acts on $\mathfrak{g}_{\geq 2}$ with a dense orbit. If DOOBS determines that B acts on $\mathfrak{g}_{\geq 2}$ with a dense orbit, then it calculates a representative of the dense orbit and a list of primes for which the result is not necessarily valid.

There is a variant of DOOBS called DOOBSLevi, see [16]. This program considers a parabolic subgroup $P = LR_u(P)$ and determines whether a Borel subgroup B_L of L acts on an ideal of $\text{Lie } R_u(P)$ with a dense orbit. The algorithm used to determine whether B_L acts on an ideal with a dense orbit is essentially the same as the DOOBS algorithm, with B_L replacing B . DOOBSLevi also records the primes for which its conclusions are not necessarily valid.

Let $e \in \mathcal{N}$ of height 3 and let λ be a cocharacter of G that is associated to e . We use the same numbering of the positive roots as in GAP4. Table 3 below lists the roots whose root subgroups together with T generate the Levi subgroup $C_G(\lambda)$ and we also list the roots whose root subspaces generate $\mathfrak{g}_{\geq 2}$ (as a B -submodule of \mathfrak{g}) for the 7 cases of height three nilpotent orbits for the simple exceptional groups, see Proposition 3.21. These are determined by means of the weighted Dynkin diagrams.

Type of G	Bala–Carter Label	Generators for L	Generators for $\mathfrak{g}_{\geq 2}$
G_2	\tilde{A}_1	α_2	α_4
F_4	$A_1 + \tilde{A}_1$	$\Pi \setminus \{\alpha_4\}$	α_{16}
E_6	$3A_1$	$\Pi \setminus \{\alpha_4\}$	α_{24}
E_7	$(3A_1)'$	$\Pi \setminus \{\alpha_3\}$	α_{37}
E_7	$4A_1$	$\Pi \setminus \{\alpha_2, \alpha_7\}$	α_{30}, α_{53}
E_8	$3A_1$	$\Pi \setminus \{\alpha_7\}$	α_{74}
E_8	$4A_1$	$\Pi \setminus \{\alpha_2\}$	α_{69}

TABLE 3. Height Three Nilpotent Orbits in the Exceptional Lie Algebras.

The height 3 cases for the exceptional groups were analyzed using the DOOBSLevi algorithm. It turns out that there are no characteristic restrictions in these cases:

Lemma 3.40. *If G is simple of exceptional type and $e \in \mathcal{N}$ with $\text{ht}(e) = 3$, then e is spherical.*

Corollaries 3.31 and 3.39 combined with Lemma 3.40 give the following result.

Proposition 3.41. *Let G be a connected reductive algebraic group and let $e \in \mathcal{N}$. If $\text{ht}(e) = 3$, then e is spherical.*

Proof. If G is simple, then the statement follows from Corollaries 3.31 and 3.39 and Lemma 3.40. In the general case we argue as in Lemma 3.4 to reduce to the simple case. \square

3.8. The Classification. Our main classification theorem now follows readily from Lemma 3.4 and Propositions 3.16 and 3.41.

Theorem 3.42. *Let G be a connected reductive algebraic group. Suppose that $\text{char } k$ is a good prime for G . Then a nilpotent element $e \in \mathfrak{g}$ is spherical if and only if $\text{ht}(e) \leq 3$.*

Remark 3.43. Let G be a simple algebraic group and let $\text{char } k$ be a good prime for G . Then the spherical nilpotent orbits are given in Table 4. We present the orbits by listing the corresponding partition in the classical cases or by giving the corresponding Bala–Carter label for the exceptional groups.

Type of G	Spherical Orbits
A_n	$[1^j, 2^i]$
B_n	$[1^j, 2^{2i}]$, or $[1^j, 2^{2i}, 3]$ with $i \geq 0$
C_n	$[1^{2j}, 2^i]$
D_n	$[1^j, 2^{2i}]$, or $[1^j, 2^{2i}, 3]$ with $i \geq 0$
G_2	A_1 or \tilde{A}_1
F_4	A_1 , \tilde{A}_1 , or $A_1 + \tilde{A}_1$
E_6	A_1 , $2A_1$, or $3A_1$
E_7	A_1 , $2A_1$, $(3A_1)'$, $(3A_1)''$, or $4A_1$
E_8	A_1 , $2A_1$, $3A_1$, or $4A_1$

TABLE 4. The spherical nilpotent Orbits for G simple.

Remark 3.44. Using the fact that in good characteristic a Springer map affords a bijection between the set of unipotent G -conjugacy classes and the set of nilpotent G -orbits (see [45]), Theorem 3.42 also gives a classification of the spherical unipotent classes in G . Here we define the height of a unipotent element u of G as the height of the image of u in \mathcal{N} under a Springer isomorphism.

4. APPLICATIONS AND COMPLEMENTS

Here we discuss some applications of the main result and some further consequences.

4.1. Spherical Distinguished Nilpotent Elements. Recall that a nilpotent element $e \in \mathcal{N}$ is distinguished in \mathfrak{g} if every torus contained in $C_G(e)$ is contained in the centre of G . For now we assume that G is simple, so e is distinguished in \mathfrak{g} if and only if any torus contained in $C_G(e)$ is trivial and hence $C_G(e)^\circ$ is unipotent. Further recall that $\kappa_G(G \cdot e) = \kappa_G(G/C_G(e)^\circ)$, cf. equation (2.15). Since $C_G(e)^\circ$ is connected and unipotent, it is contained in the unipotent

radical U of a Borel subgroup $B = TU$ of G . Let $B^- = TU^-$ be the unique opposite Borel subgroup to $B = TU$ relative to T , see [20, §26.2]. Consequently, $B^- \cap C_G(e)^\circ \subseteq B^- \cap U = \{1\}$. Thus, by equation (2.11), we have $\kappa_G(G/C_G(e)^\circ) = \dim G - \dim C_G(e)^\circ - \dim B^- = \dim U - \dim C_G(e)$, or equivalently, $\kappa_G(G \cdot e) = |\Psi^+| - \dim C_G(e)$. We summarize what we have just shown.

Proposition 4.1. *Let $e \in \mathcal{N}$ be a distinguished nilpotent element. Then*

$$\kappa_G(G \cdot e) = |\Psi^+| - \dim C_G(e).$$

Remark 4.2. Proposition 4.1 was first observed by Panyushev for a field of characteristic zero in [32, Cor. 2.4].

If G is a simple classical group, then the distinguished nilpotent elements are given as follows, see Lemmas 4.1 and 4.2 in [22].

Lemma 4.3. *Let $e \in \mathcal{N}$ and let π_e be the corresponding partition of $\dim V$.*

- (i) *If $G = \mathrm{SL}(V)$, then e is distinguished if and only if $\pi_e = [\dim V]$.*
- (ii) *If $G = \mathrm{Sp}(V)$, then e is distinguished if and only if π_e consists only of distinct even parts.*
- (iii) *If $G = \mathrm{SO}(V)$, then e is distinguished if and only if π_e consists only of distinct odd parts.*

Corollary 4.4. *If $G = \mathrm{SO}(V)$ and $e \in \mathcal{N}$ is spherical and distinguished, then $\mathrm{ht}(e) = 2$.*

Proof. Thanks to Proposition 3.21, the height 3 nilpotent elements have partitions of the form $\pi = [1^s, 2^{2r}, 3]$, where $r > 0$. Thus such a partition has even parts and so is not distinguished. So if e is spherical and distinguished, then $\mathrm{ht}(e) = 2$. \square

Proposition 2.27 and Lemma 4.3 imply the following result.

Proposition 4.5. *Let $e \in \mathcal{N}$ be distinguished and π_e be the corresponding partition of $\dim V$.*

- (i) *If $G = \mathrm{SL}(V)$, then $\mathrm{ht}(e) = 2$ if and only if $\pi_e = [2]$.*
- (ii) *If $G = \mathrm{Sp}(V)$, then $\mathrm{ht}(e) = 2$ if and only if $\pi_e = [2]$.*
- (iii) *If $G = \mathrm{SO}(V)$, then $\mathrm{ht}(e) = 2$ if and only if $\pi_e = [3]$ or $\pi_e = [1, 3]$.*

Theorem 4.6. *If G is a simple algebraic group and $e \in \mathcal{N}$ is spherical and distinguished, then G is of type A_1 .*

Proof. For G simple classical, Proposition 4.5 implies that G is of type A_1 . For G of exceptional type it follows from Remark 3.43 and the tables in [10, §13] that there are no nilpotent orbits in \mathfrak{g} that are both spherical and distinguished. \square

4.2. Orthogonal Simple Roots and Spherical Nilpotent Orbits. In [33, Thm. 3.4], Panyushev proved that if the characteristic of k is zero, then $e \in \mathcal{N}$ is spherical if and only if there exist pairwise orthogonal simple roots $\alpha_1, \alpha_2, \dots, \alpha_t$ in Π such that $G \cdot e$ contains an element of the form $\sum_{i=1}^t e_{\alpha_i}$ where $e_{\alpha_i} \in \mathfrak{g}_{\alpha_i} \setminus \{0\}$. By pairwise orthogonal we mean that $\langle \alpha_i, \alpha_j \rangle = 0$ for $i \neq j$. In this subsection we show that this is also the case if the characteristic of k is good for G .

Lemma 4.7. *Let DG be of type A_1^t for some $t \geq 1$. Then there is precisely one distinguished nilpotent orbit in \mathcal{N} .*

Proof. Since the nilpotent orbits of G in \mathfrak{g} are precisely the nilpotent orbits of $\mathcal{D}G$ in $\text{Lie } \mathcal{D}G$, we may assume that G is semisimple. Thus, $G = G_1 G_2 \cdots G_r$ and each G_i is of type A_1 . There is precisely one distinguished nilpotent orbit when G_i is of type A_1 : the unique non-zero nilpotent orbit. Also $G \cdot e$ is distinguished in \mathfrak{g} if and only if $G_i \cdot e_i$ is distinguished in $\mathfrak{g}_i = \text{Lie } G_i$ for all i , where $e = e_1 + \cdots + e_r$ and $e_i \in \mathfrak{g}_i$ is nilpotent. \square

Lemma 4.8. *Let $e \in \mathcal{N}$ and S be a maximal torus of $C_G(e)$. Then $\mathcal{D}C_G(S)$ is of type A_1^t for some $t \geq 1$ if and only if there exist pairwise orthogonal simple roots $\alpha_1, \alpha_2, \dots, \alpha_t$ in Π such that $G \cdot e$ contains an element of the form $\sum_{i=1}^t e_{\alpha_i}$, where $e_{\alpha_i} \in \mathfrak{g}_{\alpha_i} \setminus \{0\}$.*

Proof. Suppose that $\mathcal{D}C_G(S)$ is of type A_1^t . Let $\alpha_1, \dots, \alpha_t$ be simple roots of Φ , where Φ is the root system of $C_G(S)$ relative to a maximal torus T of $C_G(S)$. As $\mathcal{D}C_G(S)$ is of type A_1^t , the roots $\alpha_1, \dots, \alpha_t$ are pairwise orthogonal. Clearly, $e \in \text{Lie } C_G(S) = \mathfrak{c}_{\mathfrak{g}}(S)$ and e is distinguished in $\mathfrak{c}_{\mathfrak{g}}(S)$, see Proposition 2.17. By Lemma 4.7, an element of the form $\sum_{i=1}^t e_{\alpha_i}$ is also distinguished in $\mathfrak{c}_{\mathfrak{g}}(S)$ and there is precisely one distinguished nilpotent orbit in $\mathfrak{c}_{\mathfrak{g}}(S)$. Thus, e and $\sum_{i=1}^t e_{\alpha_i}$ are in the same $C_G(S)$ -orbit, hence they are in the same G -orbit. So $G \cdot e$ contains an element of the desired form.

Conversely, suppose that there exist pairwise orthogonal simple roots $\alpha_1, \alpha_2, \dots, \alpha_t \in \Psi$ such that $G \cdot e$ contains an element of the form $e' = \sum_{i=1}^t e_{\alpha_i}$. Let H be the subgroup of G generated by $\{T, U_{\pm \alpha_i} \mid 1 \leq i \leq t\}$, where T is as in the previous paragraph. Then $\mathcal{D}H$ is of type A_1^t . By construction, e' is distinguished in \mathfrak{h} . By Proposition 2.17, H is of the form $C_G(S')$, where S' is a maximal torus of $C_G(e')$. Thus, $\mathcal{D}C_G(S')$ is of type A_1^t . Since e and e' are G -conjugate, so are $C_G(e)$ and $C_G(e')$, as well as S and S' . Finally, we get that $C_G(S)$ and $C_G(S')$ are G -conjugate. The result follows. \square

Lemma 4.9. *If $e \in \mathcal{N}$ is spherical, then $\mathcal{D}C_G(S)$ is of type A_1^t for some $t \geq 1$.*

Proof. Let λ be a cocharacter of $G_G(S)$ that is associated to e , i.e. $\lambda \in \Omega_{C_G(S)}^a(e)$. Then, since $\text{Lie } C_G(S) = \mathfrak{c}_{\mathfrak{g}}(S)$, it follows from [11, Cor. 3.21] that $\lambda \in \Omega_G^a(e)$. As e is spherical in \mathfrak{g} , we have $\text{ht}(e) \leq 3$, by Theorem 3.42. As $\lambda \in \Omega_{C_G(S)}^a(e)$, we also have $\text{ht}(e) \leq 3$ when we regard e as an element of $\mathfrak{c}_{\mathfrak{g}}(S)$. Thus, again by Theorem 3.42, e is spherical in $\mathfrak{c}_{\mathfrak{g}}(S)$. So e is distinguished and spherical in $\mathfrak{c}_{\mathfrak{g}}(S)$ and so $\mathcal{D}C_G(S)$ is of type A_1^t , by Theorem 4.6. \square

In order to prove the reverse implication of Lemma 4.9 we first need to consider the group $C_G(S)$. If G is classical, then the structure of $C_G(S)$ can be determined from the partition π_e corresponding to e ; see [22, §4.8] for the following result.

Lemma 4.10. *Let G be simple classical and $e \in \mathcal{N}$ with corresponding partition π_e .*

- (i) *If G is of type A_n and $\pi_e = [1^{r_1}, 2^{r_2}, \dots]$, then $\mathcal{D}C_G(S)$ is of type $\prod_{i \geq 1} A_{i-1}^{r_i}$.*
- (ii) *If G is of type B_n and $\pi_e = [1^{2s_1+\epsilon_1}, 2^{2s_2}, 3^{2s_3+\epsilon_3}, \dots]$, where $s_i \geq 0$ and $\epsilon_i \in \{0, 1\}$, then $\mathcal{D}C_G(S)$ is of type $\prod_{i \geq 1} A_{i-1}^{s_i} \times B_m$, where $2m+1 = \sum_{\epsilon_i \neq 0} i$.*
- (iii) *If G is of type C_n and $\pi_e = [1^{2s_1}, 2^{2s_2+\epsilon_2}, 3^{2s_3}, 4^{2s_4+\epsilon_4}, \dots]$, where $s_i \geq 0$ and $\epsilon_i \in \{0, 1\}$, then $\mathcal{D}C_G(S)$ is of type $\prod_{i \geq 1} A_{i-1}^{s_i} \times C_m$, where $2m = \sum_{\epsilon_i \neq 0} i$.*
- (iv) *If G is of type D_n and $\pi_e = [1^{2s_1+\epsilon_1}, 2^{2s_2}, 3^{2s_3+\epsilon_3}, \dots]$, where $s_i \geq 0$ and $\epsilon_i \in \{0, 1\}$, then $\mathcal{D}C_G(S)$ is of type $\prod_{i \geq 1} A_{i-1}^{s_i} \times D_m$, where $2m = \sum_{\epsilon_i \neq 0} i$.*

Lemma 4.11. *If G is simple classical and $\mathcal{D}C_G(S)$ is of type A_1^t , then e is spherical.*

Proof. First suppose that G is of type A_n . Since $\mathcal{DC}_G(S)$ is of type A_1^t , it follows from Lemma 4.10 that $r_i = 0$ for all $i \geq 3$. Thus $\pi_e = [1^{r_1}, 2^{r_2}]$ and so e is spherical, by Remark 3.43.

Let G be of type B_n . Since $\mathcal{DC}_G(S)$ is of type A_1^t , it follows from Lemma 4.10 that $s_i = 0$ for $i \geq 3$ and $m \leq 1$, so $2m + 1 \leq 3$. Since $2m + 1$ is a sum of distinct odd integers, we either have $2m + 1 = 1$ or $2m + 1 = 3$. Thus $\pi_e = [1^{2s_1+1}, 2^{2s_2}]$ or $\pi_e = [1^{2s_1}, 2^{2s_2}, 3]$ and so e is spherical, again by Remark 3.43.

Let G be of type C_n . Since $\mathcal{DC}_G(S)$ is of type A_1^t , it follows from Lemma 4.10 that $s_i = 0$ for $i \geq 3$ and $m \leq 1$, so $2m \leq 2$. Since $2m$ is a sum of distinct even integers, we either have $2m = 0$ or $2m = 2$. Thus $\pi_e = [1^{2s_1}, 2^{2s_2}]$ or $\pi_e = [1^{2s_1}, 2^{2s_2+1}]$ and so, by Remark 3.43, e is spherical.

Finally, let G be of type D_n . Since $\mathcal{DC}_G(S)$ is of type A_1^t , it again follows from Lemma 4.10 that $s_i = 0$ for $i \geq 3$ and $m \leq 2$, so $2m \leq 4$. Since $2m$ is a sum of distinct odd integers, we either have $2m = 0$ or $2m = 1 + 3$. Thus $\pi_e = [1^{2s_1}, 2^{2s_2}]$ or $\pi_e = [1^{2s_1+1}, 2^{2s_2}, 3]$ and so, by Remark 3.43, e is spherical. \square

All that remains is to check the exceptional cases. The Bala–Carter label of $e \in \mathcal{N}$ gives the Dynkin type of a Levi subgroup L of G such that e is distinguished in $\text{Lie } \mathcal{DL}$. By Proposition 2.17, such a Levi subgroup is the centralizer of a maximal torus of $C_G(e)$. Thus, the Bala–Carter label gives the type of $\mathcal{DC}_G(S)$. It follows from the tables in [10, §13] and Remark 3.43 that any nilpotent orbit with Bala–Carter label A_1^t is spherical. We summarize this in Table 5 below.

Type	Bala–Carter Label	Height	Type	Bala–Carter Label	Height
G_2	A_1	2	E_7	A_1	2
G_2	\tilde{A}_1	3	E_7	$2A_1$	2
F_4	A_1	2	E_7	$(3A_1)''$	2
F_4	\tilde{A}_1	2	E_7	$(3A_1)'$	3
F_4	$A_1 + \tilde{A}_1$	3	E_7	$4A_1$	3
E_6	A_1	2	E_8	A_1	2
E_6	$2A_1$	2	E_8	$2A_1$	2
E_6	$3A_1$	3	E_8	$3A_1$	3
-	-	-	E_8	$4A_1$	3

TABLE 5. Orbits in Exceptional Lie Algebras with $\mathcal{DC}_G(S)$ of Type A_1^t .

Lemma 4.12. *If G is a simple exceptional algebraic group and $\mathcal{DC}_G(S)$ is of type A_1^t , then e is spherical.*

Lemma 4.13. *Let $e \in \mathcal{N}$. If $\mathcal{DC}_G(S)$ is of type A_1^t , then $e \in \mathfrak{g}$ is spherical.*

Proof. For G simple, the result follows from Lemmas 4.11 and 4.12. In the general case let $\mathcal{DG} = G_1 G_2 \cdots G_r$ be a commuting product of simple groups and $e = e_1 + e_2 + \cdots + e_r$, where $e_i \in \mathfrak{g}_i = \text{Lie } G_i$ and each e_i is nilpotent. A maximal torus S of $C_G(e)$ is of the form

$S_1 S_2 \cdots S_r$, where S_i is a maximal torus of $C_{G_i}(e_i)$. The simple case implies that $\mathcal{DC}_{G_i}(S_i)$ is of type A_1^t . \square

Lemmas 4.13 and 4.8 now imply the main result of this subsection.

Theorem 4.14. *Let $e \in \mathcal{N}$ and let S be a maximal torus of $C_G(e)$. Then the following are equivalent.*

- (i) e is spherical;
- (ii) $\mathcal{DC}_G(S)$ is of type A_1^t ;
- (iii) there exist pairwise orthogonal simple roots $\alpha_1, \alpha_2, \dots, \alpha_t \in \Pi$ such that $G \cdot e$ contains an element of the form $\sum_{i=1}^t e_{\alpha_i}$, where $e_{\alpha_i} \in \mathfrak{g}_{\alpha_i} \setminus \{0\}$.

4.3. Spherical Orbits and ad-Nilpotent Ideals. In this section we generalize some results from [34] and [35] to a field of good characteristic.

When G is simple and classical, Panyushev gave simple algebraic criteria for a nilpotent element $e \in \mathcal{N}$ to be spherical in [32, §4]. We show that these criteria are still valid for a field of good characteristic.

Lemma 4.15. *Let G be a simple classical algebraic group and $e \in \mathcal{N}$.*

- (i) *Let e be a nilpotent matrix in \mathfrak{sl}_n or \mathfrak{sp}_n . Then e is spherical if and only if $e^2 = 0$.*
- (ii) *Let e be a nilpotent matrix in \mathfrak{so}_n . Then e is spherical if and only if the rank of e^2 is at most one.*

Proof. Let e be a nilpotent matrix in \mathfrak{sl}_n or \mathfrak{sp}_n . If e is spherical, then $\pi_e = [1^j, 2^i]$, for appropriate i and j , see Remark 3.43. By considering the corresponding Jordan blocks for π_e , we see that $e^2 = 0$. Conversely, if $e^2 = 0$, then e is conjugate to an element e' with partition $\pi_{e'} = [1^j, 2^i]$ and so e is spherical, again by Remark 3.43.

Let e be a nilpotent matrix in \mathfrak{so}_n . If e is spherical, then $\pi_e = [1^j, 2^i]$ or $\pi_e = [1^j, 2^i, 3]$, for appropriate i and j , see Remark 3.43. By considering the corresponding Jordan blocks for π_e , we see that either $e^2 = 0$ or e^2 has partition $\pi_{e^2} = [1^k, 2]$. Thus the rank of e^2 is either 0 or 1. Conversely, if the rank of e^2 is at most 1, then e is conjugate to an element e' with partition $\pi_{e'} = [1^j, 2^i]$ or $\pi_{e'} = [1^j, 2^i, 3]$ and so e is spherical, again by Remark 3.43. \square

In [34] and [35], D.I. Panyushev and the second author gave a classification of the spherical ideals of $\mathfrak{b} = \text{Lie } B$ contained in $\mathfrak{b}_u = \text{Lie } R_u(B)$, where B is a Borel subgroup of G in characteristic 0. An ideal \mathfrak{c} of \mathfrak{b} is *ad-nilpotent* if \mathfrak{c} is contained in \mathfrak{b}_u . An ad-nilpotent ideal \mathfrak{c} of \mathfrak{b} is called *spherical* if its G -saturation $G \cdot \mathfrak{c} = \{x \cdot e \mid x \in G, e \in \mathfrak{c}\}$ is a spherical G -variety. First in [34, Cor. 2.4] it is proved that if \mathfrak{a} is an Abelian ideal of \mathfrak{b} , then \mathfrak{a} is spherical. In [35, Prop. 4.1 and Thm. 4.2] it is proved that there are non-abelian spherical ideals only if G is not simply-laced, that is if the Dynkin diagram of G has a multiple bond.

Theorem 2.3 in [34] states that any G -orbit meeting an abelian ad-nilpotent ideal \mathfrak{a} is spherical. This is proved by means of the fact that an orbit $G \cdot e$ is spherical if and only if $\text{ad}(e)^4 = 0$, see [32, Cor. 2.2]. Unfortunately, this equivalence is no longer true in positive characteristic, see Example 4.17. However, the forward implication of this equivalence is still valid in good characteristic.

Lemma 4.16. *If $e \in \mathcal{N}$ is spherical, then $\text{ad}(e)^4 = 0$.*

Proof. If e is spherical, then by Theorem 3.42 $\text{ht}(e) \leq 3$. Let $\mathfrak{g} = \bigoplus_{i=-3}^3 \mathfrak{g}(i)$ be the grading of \mathfrak{g} afforded by an associated cocharacter in $\Omega_G^a(e)$. We have that $e \in \mathfrak{g}(2)$. Consequently, $\text{ad}(e)^4(\mathfrak{g}(i)) \subseteq \mathfrak{g}(i+8) = \{0\}$ for any $-3 \leq i \leq 3$. Consequently, $\text{ad}(e)^4 = 0$ on all of \mathfrak{g} . \square

The next example shows that the converse of Lemma 4.16 is not true in general in positive characteristic.

Example 4.17. Let $G = \text{SL}_3(k)$ and $\text{char } k = 3$. So $\mathfrak{g} = \mathfrak{sl}_3(k)$. Set $e = e_{2,1} + e_{3,2}$, where $e_{i,j}$ is the elementary matrix with a 1 in the (i,j) position and 0's elsewhere. So e is a regular nilpotent element in \mathfrak{g} . Consider the grading of \mathfrak{g} afforded by an associated cocharacter in $\Omega_G^a(e)$. We have $\mathfrak{g} = \bigoplus_{i=-2}^2 \mathfrak{g}(2i)$. In order to prove $\text{ad}(e)^4 = 0$, it is sufficient to show that $\text{ad}(e)^4(\mathfrak{g}(-4)) = \{0\}$. Clearly, $\mathfrak{g}(-4) = ke_{1,3}$. Now $\text{ad}(e)(e_{1,3}) = e_{2,3} - e_{1,2}$ and $\text{ad}(e)(e_{2,3} - e_{1,2}) = e_{1,1} - 2e_{2,2} + e_{3,3}$. Since $\text{char } k = 3$, we have $e_{1,1} - 2e_{2,2} + e_{3,3} = e_{1,1} + e_{2,2} + e_{3,3}$ and $e_{1,1} + e_{2,2} + e_{3,3} \in Z(\mathfrak{g})$. Thus, $\text{ad}(e)^4 = 0$. However, e is not spherical, as $\pi_e = [3]$, see Remark 3.43.

We note that Proposition 4.1 and Theorem 4.2 in [35] both also hold in good characteristic, as their proofs only require properties of the underlying root system Ψ and the results established in Lemmas 4.15 and 4.16.

So we are left to show that if \mathfrak{a} is an abelian ad-nilpotent ideal, then \mathfrak{a} is spherical. Since $G \cdot \mathfrak{a}$ is irreducible, it is the closure of some nilpotent orbit, say $\overline{G \cdot e} = G \cdot \mathfrak{a}$. The maximal abelian ad-nilpotent ideals of \mathfrak{b} are the same in good characteristic as in characteristic zero, see Table 1 in [42] and Tables I and II in [34, §4]. Using the description of the orbits in Tables I and II in [34, §4], we infer that the Bala–Carter label of $G \cdot e$ is of the form A_1^t , so $G \cdot e$ is spherical, thanks to Theorem 4.14. Since $G \cdot e$ is open in $G \cdot \mathfrak{a}$, it follows that $G \cdot \mathfrak{a}$ is spherical. It is straightforward to get the sphericity of $G \cdot \mathfrak{a}$ for any abelian ideal \mathfrak{a} of \mathfrak{b} from the sphericity result of the maximal abelian ideals. Thus we have established the following.

Theorem 4.18. *Let \mathfrak{a} be an abelian ad-nilpotent ideal of \mathfrak{b} . Then \mathfrak{a} is spherical.*

As a corollary of Theorem 4.18 we get [42, Thm. 1.1] in good characteristic.

Corollary 4.19. *Let P be a parabolic subgroup of G and let \mathfrak{a} be an abelian ideal of $\text{Lie } P$ in $\text{Lie } R_u(P)$. Then P acts on \mathfrak{a} with finitely many orbits.*

Remark 4.20. We note that Theorem 4.18 and Corollary 4.19 do in fact hold in arbitrary characteristic, cf. [42, Thm. 1.1].

Remark 4.21. If \mathfrak{c} is a spherical ideal of \mathfrak{b} , then clearly B acts on \mathfrak{c} with a finite number of orbits. However, the converse does not hold. There are many additional instances when B acts on a given ideal \mathfrak{c} of \mathfrak{b} only with a finite number of orbits, e.g. see the results in [17] and [23].

4.4. A Geometric Characterization of Spherical Orbits. In this subsection we describe a formula characterizing spherical G -orbits in a simple algebraic group G in terms of elements of the Weyl group W of G that is proved in [8, Thm. 1]. For $x \in G$ the conjugacy class $G \cdot x$ is spherical if $G \cdot x$ is a spherical variety. While this characterization in *loc. cit.* is based on case by case arguments, recently, G. Carnovale [9, Thm. 2] gave a proof of this result which is free of case by case considerations and applies in good odd characteristic. Using the arguments from [8] combined with our classification of the spherical unipotent nilpotent orbits, Remark 3.44, we can generalize this formula to good characteristic.

Let G be simple and suppose that p is good for G . Fix a Borel subgroup B of G . Let W be the Weyl group of G and let BwB be the (B, B) -double coset of G containing $w \in W$. The following was shown in [8] in an argument independent of the characteristic of the underlying field: Suppose that \mathcal{O} is a conjugacy class in G which intersects the double coset BwB so that $\dim \mathcal{O} = \ell(w) + \text{rk}(1 - w)$ holds. Then \mathcal{O} is spherical. Here $\text{rk}(1 - w)$ denotes the rank of the linear map $1 - w$ in the standard representation of W and ℓ is the usual length function of W with respect to a distinguished set of generators of W . Conversely, let \mathcal{O} be a spherical conjugacy class in G and let BwB be the (B, B) -double coset containing the dense B -orbit in \mathcal{O} . Then $\dim \mathcal{O} = \ell(w) + \text{rk}(1 - w)$, see [9, Thm. 2]. Consequently, this gives a geometric characterization of the spherical conjugacy classes in G . For proofs we refer the reader to [8] and [9]. Observe that as a consequence of the finiteness of the Bruhat decomposition of G and the fact that any (B, B) -double coset and any conjugacy class of G are irreducible subvarieties of G , for a given conjugacy class \mathcal{O} in G there is a unique $w \in W$ such that $\mathcal{O} \cap BwB$ is dense in \mathcal{O} .

Theorem 4.22. ([8, Thm. 1]) *Let \mathcal{O} be a conjugacy class in G and let $w \in W$ be such that $\mathcal{O} \cap BwB$ is dense in \mathcal{O} . Then \mathcal{O} is spherical if and only if $\dim \mathcal{O} = \ell(w) + \text{rk}(1 - w)$.*

4.5. Bad Primes and Spherical Nilpotent Orbits. Finally, we briefly discuss the situation when the characteristic of k is bad for G . In this case the classification of the nilpotent orbits in \mathcal{N} is different from that in good characteristic, see [10, §5.11]. However, there is still only a finite number of nilpotent orbits, [18]. Unfortunately, our methods do not allow us to give a classification of the spherical nilpotent orbits in this case. For, in our classification we made use of the height of a nilpotent orbit, where the height is defined via an associated cocharacter. However, it is not known whether associated cocharacters always exist for all nilpotent elements in bad characteristic, cf. [22, §5.14, §5.15].

In principle one can still determine whether a given nilpotent orbit is spherical by a case by case analysis. Next we give two examples of this. In particular, we show that Theorem 4.14 fails in bad characteristic in general. These examples show that there can be additional spherical nilpotent orbits in bad characteristic.

Examples 4.23. (i). Let G be of type B_2 and $\text{char } k = 2$. Let α and β be the simple roots of Ψ with α the long root. Let $e = e_{\alpha+\beta} + e_{\alpha+2\beta}$. According to [22, §5.14] the centralizer $C_G(e)$ is the unipotent radical of a Borel subgroup of G . Thus, by Lemma 2.12, $C_G(e)$ is a spherical subgroup of G and so e is spherical. Note that the G -orbit of e does not contain an element of the form e_α or e_β , but e is still spherical. Thus, Theorem 4.14 is no longer true in bad characteristic. Moreover, e is distinguished in \mathfrak{g} , [22, §5.14]. This shows that Theorem 4.6 can also fail for bad characteristic.

(ii). Let G be of type G_2 and $\text{char } k = 3$. Let α and β be the simple roots of Ψ with α the long root. Let $e = e_{\alpha+2\beta} + e_{2\alpha+3\beta}$. According to [22, §5.15], the centralizer $C_G(e)$ is the unipotent radical of a Borel subgroup of G . Thus, by Lemma 2.12, $C_G(e)$ is a spherical subgroup of G and so e is spherical. Again, the G -orbit of e does not contain an element of the form e_α or e_β , but e is spherical. Again, e is distinguished in \mathfrak{g} , [22, §5.15].

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